AFOSR-1017

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. BR-33

Some Uniqueness Theorems for the Reduced Wave Equation

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Mathematics Division
Air Force Office of Scientific Research
Contract No. AF 49(638)-229
Project No. 47500

JUNE 1961

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Abstract

This paper deals with various extensions of the Magnus-Rellich uniqueness theorem for the reduced wave equation in infinite domains. The theorem is extended to cover piecewise smooth boundary surfaces of a general kind, and mixed boundary conditions; no auxiliary "edge conditions" are required at edges or at discontinuities in the boundary conditions - continuity of the wave function in the closure of the domain is sufficient. Another extension treats infinite boundaries; for real values of the propagation constant, these are restricted to surfaces which are (generalized) cones sufficiently far from the origin.

Table of Contents

1. Introduction.

The purpose of this paper is to complete and extend in a number of ways the Magnus-Rellich uniqueness theorem for the reduced wave equation in infinite domains. The theorem was stated by Rellich [1] essentially as follows:

Let G be the exterior of a finite closed surface B. There exists at most one function $u = u(x_1, x_2, \ldots, x_n)$ defined in $\overline{G} = G \cup B$ such that

(a) $u \in C^{(2)}$ in G

(b) $\Delta u + k^2 u = 0 \qquad (k > 0)$

(c) u assumes given values on B

(d) $\lim_{r\to\infty} r^{\frac{n-1}{2}} (\partial u/\partial r - iku) = 0$

uniformly along all rays from the origin. $\left[r = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2} \right]$

(e) B and u satisfy such regularity conditions as to ensure the validity of the following case of Green's formula:

$$\int_{G_\rho} \left[\overline{w}(\Delta w + k^2 w) - w(\Delta \overline{w} + k^2 \overline{w}) \right] dV = \int_{r=\rho} (\overline{w} w_r - w \overline{w}_r) dS \qquad (1.1)$$

Here $w = u - v$, where v is any other solution which satisfies the same regularity conditions as u, as well as condition (d), and assumes the same values on B. G_ρ is the domain exterior to B and interior to the hypersphere with center at the origin and radius ρ. The radius ρ is so large that the hypersphere contains B in its interior.

We shall show, in the three dimensional case, that (e) may be replaced by the simple conditions that u be a solution of the wave equation (b) which is continuous in $\overline{G}$, and that B belong to a certain fairly general class of piecewise smooth closed surfaces (herein designated "regular closed surfaces".) No auxiliary

"edge conditions" will be required of u at edges or vertices. Moreover, we shall extend the theorem to include the mixed boundary value problem; i.e., instead of condition (c), u may be specified on some parts of B, and $\partial u/\partial n + \beta u$ specified over the remainder of B, β being in general a (possibly discontinuous) function of position on B.

The remaining conditions of the above theorem will also be generalized or deleted. It will be proved that condition (a) is superfluous, being deducible from the continuity of u and condition (b). In condition (b) we shall allow k to be any complex number other than zero satisfying $\operatorname{Im} k \geqq 0$, $\operatorname{Re} k \geqq 0$. The radiation condition (d) will be replaced by the weaker integral condition

$$\lim_{\rho \to \infty} \int_{r=\rho} \left|\partial u/\partial r - iku\right|^2 dS = 0. \tag{1.2}$$

The resulting statement, Theorem 2.1 below, is still not general enough to cover a number of problems of interest in diffraction theory, for example, the problem of diffraction by a semi-infinite circular cone. Besides being infinite, this surface possesses a singularity - its apex - which is not permitted by the hypothesis of Theorem 2.1. Moreover, the restriction to closed surfaces rules out the application of the theorem to such situations as diffraction by a disk of zero thickness. The latter restriction is not hard to remove, however, and the theorem will first be extended to admit a considerably more general class of finite boundaries. It will then be further extended to include infinite boundaries, and boundaries with singularities of the type occurring at the apex of a (generalized) cone. When k is a real number, the extension to infinite boundaries is limited to the case where the domain is "eventually conical", i.e. outside a sufficiently large sphere, it is generated by a ray emanating from the center.

Furthermore, it will be assumed in this case that sufficiently far from the center only the Dirichlet or Neumann boundary conditions obtain, possibly changing from one to the other along a generatrix of the cone. In the case of domains with infinite boundaries, the integral form of the radiation condition (1.2) will also be used, the region of integration being that part of the sphere of radius ρ contained in the domain.

The original uniqueness theorem for the reduced wave equation in infinite domains was stated by Sommerfeld [3]; along with the radiation condition an additional "finiteness condition" was assumed:

$$u = O(r^{-1}), \qquad \text{as } r \longrightarrow \infty \tag{1.3}$$

uniformly with respect to direction. The theorem was first proved in this form by Magnus [4]. The elimination of the extraneous condition (1.3) was accomplished shortly afterwards by Rellich, resulting in the theorem stated above. Magnus dealt with the Dirichlet problem for smooth closed boundaries, and assumed the existence and continuity of $\partial u/\partial n$ on the boundary. As seen above, Rellich also restricted himself to the Dirichlet problem, but did not give explicit conditions for the boundary surface, or for the behavior of the wave function at the boundary. Instead, he gave the complicated implicit condition (e). Both authors assumed k to be real. This restriction was removed by several writers, of whom Atkinson [5] appears to have been the first.

Diffraction at edges has been discussed by a number of writers, among whom are Meixner [6], Bouwkamp [7] and Peters and Stoker [8]; in every case auxiliary edge conditions were assumed. Meixner proved uniqueness for the electromagnetic problem using a finite energy condition, as well as the assumption that the field could be expanded in the vicinity of the edge in a series of positive

fractional powers of r (where r is the distance to the edge). He stated, without proof or references, that in the acoustic case, boundedness of the solution is a sufficient condition for uniqueness. Bouwkamp considered diffraction by an aperture in an acoustically hard or soft screen; he assumed boundedness and a finite energy condition in the vicinity of an edge, conjecturing that boundedness alone might be sufficient. Peters and Stoker (who considered the two dimensional problem) assumed a condition at a corner of the form

$$|\nabla u| = O(r^{-\alpha}), \quad \text{as } r \rightarrow 0 \; ; \; 0 < \alpha < 1. \tag{1.4}$$

where r is the distance to the corner. Wilcox [9] apparently proved uniqueness for the case where the boundary surface is "regular" in the sense of Kellogg [10], having edges and vertices, but he effectively ruled out such singularities of the surface by requiring that the solution be of class $C^{(2)}(\overline{G})$

The general mixed boundary value problem in diffraction theory has apparently not been considered elsewhere, although some special problems have been investigated. The third boundary value problem (for smooth surfaces) was treated by Leis [12].

Regarding infinite boundaries, Peters and Stoker [8] proved uniqueness for the case of a boundary in two dimensions which consists, eventually, of two straight lines extending to infinity. For the case of real k, the extension of the uniqueness theorem to infinite boundaries presented here treats the three dimensional analogues of the infinite boundaries of [8], namely, eventually conical boundaries. However, in the three dimensional case, where the cones have arbitrary shapes (e.g., they may have edges extending to infinity) and where mixed boundary conditions are considered, the proof, although along lines similar to that of [8], involves several interesting points not encountered in the two dimensional problem Finally, it should be mentioned that Rellich [2] discussed certain infinite domains in which the solution is not unique.

The generality of the results presented here is made possible in large measure by certain estimates "up to the boundary" for the derivatives of solutions of second order elliptic equations. These estimates are due to Schauder [13] for the Dirichlet problem, and to Miranda [14] and Agmon, Douglis and Nirenberg [16] for general boundary conditions.

The plan of the paper is as follows. In Section 2 we define some terms and give a complete statement of the uniqueness theorem for the exterior of a regular closed surface. In Sections 3 and 4 we deal with the behavior near the boundary of solutions of the wave equation. Essentially, the purpose is to enable us to derive a formula, (Lemma 5.1), similar to (1.1)* for solutions u which satisfy homogeneous (mixed) boundary conditions on a regular closed surface. In Section 3 we first establish a certain degree of smoothness up to the boundary of u at regular boundary points, using the limited assumptions of the uniqueness theorem. This then allows us to use the "Schauder boundary estimates" (we shall use this designation regardless of which boundary condition is involved) to obtain estimates for $|\nabla u|$ near regular boundary points. In Section 4 we use these results to prove Theorem 4.1, which concerns the behavior of u near edges or discontinuities in the boundary conditions; essentially, it furnishes an estimate for $|\nabla u|$ something like (1.4). It is this theorem which makes it possible to dispense with accessory edge conditions. In Section 5 we prove the uniqueness theorem for the exterior of a regular closed surface. In Section 6 the extension to a more general class of domains with finite boundaries is carried out. Section 7 is concerned with the extension to infinite boundaries and to boundaries with conical singularities. In Section 8 we discuss several supplementary topics. These include the behavior for

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* Lemma 5.1 corresponds to the first (asymmetric) form of Green's theorem, rather than the second form used to obtain (1.1); the first form is more convenient when dealing with complex k.

large r of a wave function about an eventually conical obstacle, and the invariance of the radiation condition under a change of origin. Appendix I consists of a proof of the statement made above in connection with condition (a) of Rellich's theorem, namely, that a solution of the wave equation which is continuous in a domain G must belong to $C^{(2)}(G)$. Appendix II contains a geometrical result used in the proof of Theorem 4.1. Appendix III contains a proof of the existence of a complete set of eigenfunctions for the Beltrami operator in general spherical surface regions (with mixed boundary conditions). In addition, some required properties of the eigenfunctions and associated eigenvalues are derived. These results are used in Sections 7 and 8.

2. Definitions; Statement of the Uniqueness Theorem for Domains Bounded by Regular Closed Surfaces.

Definition 1. Let R be a region* of euclidean n-space, E^n. A complex-valued function u on R is in $C^{(m)}(R)$ for integral $m \geq 0$, if

(a) For $0 \leq k \leq m$, the k^{th} order partial derivatives, $D^k u$, exist and are continuous in $\overset{o}{R}$,** $D^o u \equiv u$ being continuous in R.

(b) For $\hat{x} \in R - \overset{o}{R}$, $\lim\limits_{\substack{x \to \hat{x} \\ x \in \overset{o}{R}}} D^k u$ exists, $o \leq k \leq m$.

It is easily seen that for $u \in C^{(m)}(R)$, if the limiting values are assigned to $D^k u$, $k \leq m$, on $R - \overset{o}{R}$, we obtain continuous extensions of the derivatives to all of R.

Definition 2. Let A be a subset of E^n. A complex-valued function u on A is said to satisfy a Hölder condition with exponent λ, $0 < \lambda < 1$, if

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* We employ the more or less standard usage: a connected open set in E^n is a domain; a region is a domain plus some (possibly all, or none) of its boundary points.

** $\dot{A}$, $\bar{A}$, and $\overset{o}{A}$ denote the boundary, closure and interior of A, respectively.

$$\sup_{x,y \in A} \frac{|u(x) - u(y)|}{|x-y|^{\lambda}} < \infty . \tag{2.1}$$

This least upper bound is called the "Hölder constant". u is said to be Hölder continuous with exponent λ in a region R if it satisfies a Hölder condition in every compact subset of R. For integral $m \geq 0$ and $0 < \lambda < 1$, $C^{(m+\lambda)}(R)$ will denote the subclass of functions in $C^{(m)}(R)$ whose derivatives of order m, when continuously extended to $\bar{R}$, are Hölder continuous with exponent λ in $\bar{R}$.*

Definition 3. A $C^{(m+\lambda)}$ surface element, (integral $m \geq 1$, $0 \leq \lambda < 1$) is a set of points in E^3 which for some system of cartesian coordinates x_1, x_2, x_3, admits a representation

$$x_3 = f(x_1, x_2) \qquad (x_1, x_2) \in \bar{D} \tag{2.2}$$

where $f \in C^{(m+\lambda)}(\bar{D})$. Here D is a domain in E^2 such that $\dot{D}$ is a rectifiable Jordan curve. In addition we require that it be possible to extend f to a function $\tilde{f}$ defined in a domain $\tilde{D}$ containing $\bar{D}$ where $\tilde{f} \in C^{(m+\lambda)}(\tilde{D})$. A point of the $C^{(m+\lambda)}$ surface element whose projection in the $x_1 - x_2$ plane lies on $\dot{D}$ is an edge point. The other points are inner points of the surface element. If F is a $C^{(m+\lambda)}$ surface element then $\overset{o}{F}$ denotes the set of its inner points.

Now let us suppose that y_1, y_2, y_3 constitute another cartesian coordinate system where the angle between the y_3 - axis and the tangent plane at any point of the surface element (2.2) is bounded away from zero. Then it can easily be shown that the surface element is also represented by the equation $y_3 = g(y_1, y_2)$ where $g \in C^{(m+\lambda)}(\bar{D}_1)$, and $\bar{D}_1$ is the projection of the surface element on the $y_1 - y_2$ plane; g and D_1 satisfy the same boundary and extensibility requirements that f and D satisfy.

Next, let F be the $C^{(m+\lambda)}$ surface element represented by (2.2) and let λ_1, $0 \leq \lambda_1 < 1$, and the integer m_1 satisfy

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* We shall often write "$u \in C^{(m+\lambda)}(R)$" without specifying λ, meaning that this is true for some λ, $0 < \lambda < 1$. We shall use a similar convention in connection with Definition 3.

$$0 \leq m_1 + \lambda_1 \leq m + \lambda \tag{2.3}$$

Then $C_S^{(m_1+\lambda_1)}(F)$, $\left[C_S^{(m_1+\lambda_1)}(\overset{o}{F})\right]$, will be the set of functions $\phi(x)$ on F, $[\overset{o}{F}]$, where $\phi(x_1,x_2, f(x_1,x_2))$ is in $C^{(m_1+\lambda_1)}(\bar{D})$, $\left[C^{(m_1+\lambda_1)}(D)\right]$. It is not difficult to show that membership of ϕ in $C_S^{(m_1+\lambda_1)}(F)$ is unaffected by a cartesian coordinate transformation, as long as F has a representation of the type (2.2) in the new coordinate system. (This would not necessarily be true if (2.3) did not hold.)

Definition 4. A subset B of E^3 is a closed surface if it is a connected, compact 2-manifold; i.e., every point of B has a neighborhood whose intersection with B is a 2-cell (a homeomorph of the interior of a circle).

A closed surface B in E^3 separates the set E^3- B into two components, one of which is unbounded and called the exterior of B. The bounded component is the interior of B. As here defined, a closed surface is always homeomorphic to a sphere with a finite number of handles.*

Definition 5. A regular closed surface is a closed surface which can be subdivided into a finite number of $C^{(2+\lambda)}$ surface elements satisfying the following condition: two adjacent surface elements must not form a " zero exterior angle" ; i.e., there shall be no points of the exterior region inside a cusp formed by two surface elements tangent at a common edge point.**

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* [17] p. 326, 141.

** If this " cusp condition" were omitted, and $C^{(2+\lambda)}$ surface elements were replaced by "regular surface elements" (Kellogg [10]), the regular closed surfaces defined here would be equivalent to those of Kellogg. (The defining functions for regular surface elements, corresponding to $f(x_1,x_2)$ in (2.2) satisfy less stringent smoothness conditions than those for $C^{(2+\lambda)}$ surface elements, having to be only continuously differentiable; on the other hand, the boundary curves of regular surface elements are smoother, consisting of a finite number of continuously differentiable arcs.)

In addition, the following notation will be adopted throughout the paper:

The open sphere of radius ρ and center x will be denoted by $\sigma(\rho,x)$; its boundary by $\Sigma(\rho,x)$.

If R is a region in E^3, and $x \in R$, then $\tau_R(\rho,x)$ is that component of $\sigma(\rho,x) \cap R$ which contains x.

$\frac{\partial u}{\partial n}$ will denote the <u>outgoing</u> normal derivative at a boundary point of a region in which u is defined. More precisely, suppose that u is defined in a region R, $\hat{x} \in R - \overset{o}{R}$, and suppose that in a neighborhood of $\hat{x}$, $\dot{R}$ is a surface possessing a tangent plane at $\hat{x}$; then $\partial u/\partial n$ is defined as

$$\lim_{\substack{x \to \hat{x} \\ x \in \ell_n \cap R}} \frac{u(\hat{x}) - u(x)}{|\hat{x} - x|}$$

if this limit exists, where ℓ_n is the straight line normal to $\dot{R}$ at $\hat{x}$. (This limit certainly exists if, as in the hypothesis on some parts of the boundary in the following theorem, $\lim\limits_{\substack{x \to \hat{x} \\ x \in \overset{o}{R}}} \nabla u$ exists*, u being defined and continuous at $\hat{x}$).

We are now in a position to state the main theorem of this paper, namely:

<u>Theorem 2.1</u>. Let G be the exterior of a regular closed surface B, and suppose that $u(x)$, defined in $\overline{G}$, satisfies the following conditions:

(a) $u \in C^{(o)}(\overline{G})$;

(b) u is a solution in G of the reduced wave equation**

$$\Delta u + k^2 u = 0, \qquad k \neq 0, \quad \operatorname{Re} k,\ \operatorname{Im} k \geq 0; \tag{2.4}$$

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* In this expression it will always be implicitly presupposed that ∇u is defined in $\sigma(\rho,\hat{x}) \cap \overset{o}{R}$ for some $\rho > 0$.

** Here we assume only that the three derivatives $\partial^2 u/\partial x_i^2$, $i=1,2,3$, exist.

(c) B has a decomposition into a finite number of $C^{(2+\lambda)}$ surface elements F which are divided into two classes * $\mathcal{S}$ and $\mathcal{H}$ as follows:

On F in $\mathcal{S}$, u satisfies the homogeneous Dirichlet boundary condition u = 0.

At every inner point $\hat{x}$ of F in $\mathcal{H}$, $\lim_{\substack{x\to\hat{x}\\ x\in G}} \nabla u$ exists, and u satisfies the boundary condition $\partial u/\partial n + \beta u = 0$, where β is a non-negative function belonging to $C_S^{(1+\lambda)}(F)$;

(d) the radiation condition

$$\lim_{\rho\to\infty} \int_{\Sigma(\rho,o)} \left| \frac{\partial u}{\partial r} - iku \right|^2 dS = 0. \tag{2.5}$$

Then, for all $x \in \overline{G}$

$$u(x) \equiv 0. \tag{2.6}$$

Next, in condition (c), let us replace the homogeneous boundary conditions u = 0, and $\partial u/\partial n + \beta u = 0$ by the corresponding inhomogeneous boundary conditions $u = \psi$ and $\partial u/\partial n + \beta u = \phi$, respectively, where ϕ and ψ are arbitrary functions on their respective surface elements. Let us call the resulting condition (c'). Then it is easy to see that if two functions satisfy conditions (a), (b), (c') and (d), their difference satisfies (a), (b), (c), and (d), and hence vanishes. (To show that the difference function satisfies (d) we simply use the triangle inequality for the norm $\| v \| = \left(\int_{\Sigma(\rho,o)} |v|^2 dS \right)^{1/2}$.) Thus Theorem 2.1 is equivalent to the uniqueness theorem that there exists at most one function u satisfying conditions (a), (b), (c') and (d).

It should be pointed out that the radiation condition (d) is invariant under a translation of the center of the spheres $\Sigma(\rho,o)$ from the origin to any finite point. This will be shown in section 8.

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* One of these classes may, of course, be empty.

3. On the behavior of solutions near regular boundary points.

Later, (in the proof of Lemma 5.1), it will be necessary to apply Green's theorem to the wave functions u and $\overline{u}$ in certain subregions of $\overline{G}_\rho$.* These subregions are obtained from $\overline{G}_\rho$ by excluding the edge points** $\mathcal{E}$ of B by means of "tubular" surfaces arbitrarily close to $\mathcal{E}$. However, the explicit smoothness conditions on u stated in the hypothesis of Theorem 2.1 are insufficient for this application of Green's theorem, and we must therefore prove additional smoothness in the interior and at regular boundary points of G for this purpose. Moreover, as was indicated at the end of Section 1, additional smoothness properties are prerequisite for the application of the "Schauder boundary estimates".

Smoothness in the interior is covered by Appendix I, where it is proved that the continuity of a solution u of (2.4) in the domain G implies that $u \in C^{(2)}(G)$, (and hence that u is analytic in G [18]). With regard to regular boundary points, the property we need is that $u \in C^{(2+\lambda)}(G \cup \overset{\circ}{F})$ for each $C^{(2+\lambda)}$ surface element F in $\mathcal{S} \cup \mathcal{H}$ (i.e. $u \in C^{(2+\lambda)}(\overline{G} - \mathcal{E})$). The proof is different for the two cases $F \in \mathcal{S}$ and $F \in \mathcal{H}$. For $F \in \mathcal{S}$ we begin by stating (without proof) a similar result for harmonic functions, and from it we derive the desired result for wave functions.***

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* The domains G_ρ are defined in condition (e) of Rellich's theorem in Section 1.

** Throughout the paper $\mathcal{E}$ will denote the set of edge points of the $C^{(2+\lambda)}$ surface elements into which B is subdivided according to hypothesis (c) of Theorem 2.1. Thus $\mathcal{E}$ includes the points at which there are discontinuities associated with the boundary conditions as well as the points of the actual edges of B.

*** Smoothness up to the boundary of solutions of general elliptic systems has been proved by Nirenberg [19] but the general results presented there require more smoothness conditions for the surface element than are required for our special case.

Theorem 3.1 (Kellogg [11]) Let G be a domain in E^3; let F be a $C^{(2+\lambda)}$ surface element contained in $\dot{G}$ such that every point of $\overset{\circ}{F}$ has a neighborhood (in E^3) which does not intersect $\dot{G} - \overset{\circ}{F}$. Suppose that u is harmonic in G and continuous in $G \cup \overset{\circ}{F}$, and that the boundary values of u on F are of class $C_S^{(m+\lambda')}(\overset{\circ}{F})$, where m = 1 or 2, $0 < \lambda' < 1$, and $m+\lambda' \leq 2 + \lambda$. Then $u \in C^{(m+\lambda')}(G \cup \overset{\circ}{F})$.*

Lemma 3.1 (Korn [21]) Let f(x) be a bounded integrable function on a bounded domain G in E^3 and define

$$w(x) = \int_G \frac{f(x')}{|x-x'|} \, dV' \tag{3.1}$$

Then $w \in C^{(1+\lambda)}(G)$ for any λ such that $0 < \lambda < 1$. If, in addition, $f(x) \in C^{(\lambda)}(G)$, then $w \in C^{(2+\lambda)}(G)$.**

Theorem 3.2 Let G and F be as in Theorem 3.1; suppose that u is in $C^{(0)}(G \cup F)$, satisfies the wave equation (2.4) in G, and vanishes on F. Then $u \in C^{(2+\lambda)}(G \cup \overset{\circ}{F})$.

Proof: For any $\hat{x} \in F$, the assumptions on F and G imply that there is a $\rho_o = \rho_o(\hat{x}) > 0$ such that the region $\tau_{G \cup \overset{\circ}{F}}(\rho_o, \hat{x})$ possesses the following properties:

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* This theorem is proved in [11] for any m = 1,2,3,..., provided that if m > 2, F is a $C^{(m+\lambda)}$ surface element. It should be pointed out that in Kellogg's statement of the theorem there are additional conditions on the surface element which are not satisfied by our $C^{(2+\lambda)}$ surface elements F; however, for every inner point x_o of F we can find a sufficiently small piece of F containing x_o which does satisfy the additional conditions, whence it is easy to show that the theorem follows as stated. See also Graves [20], where a gap in Kellogg's proof is filled.

** The statement obtained by replacing G by $\overline{G}$ throughout the lemma is what is actually proved in [21]; moreover, there G is assumed to be the interior of a regular closed surface without edges (i.e. for every $\hat{x} \in \dot{G}$ there is a subdivision of $\dot{G}$ into $C^{(2+\lambda)}$ surface elements F such that $\hat{x}$ is an inner point of some F.) However, upon applying the results of [21] to arbitrary spheres whose closures are in G, we easily obtain the lemma as stated.

a) $\tau_{G\cup\overset{\circ}{F}}(\rho_o,\hat{x}) = \sigma(\rho_o,\hat{x}) \cap (G\cup\overset{\circ}{F})$;

b) $\dot{\tau}_{G\cup\overset{\circ}{F}}(\rho_o,\hat{x})$ is contained in $G\cup\overset{\circ}{F}$, and consists of a region on the surface $\Sigma(\rho_o,\hat{x})$ and a $C^{(2+\lambda)}$ surface element $F_1 \subset \overset{\circ}{F}$.

We shall prove that $u \in C^{(2+\lambda)}(\tau)$ for any region $\tau = \tau_{G\cup\overset{\circ}{F}}(\rho_o,\hat{x})$, from which the theorem will follow by a simple argument.

First, we define a continuous extension $\tilde{u}$ of u over the entire sphere $\sigma = \sigma(\rho_o,\hat{x})$ by setting

$$\tilde{u}(x) = \begin{cases} u(x), & x \in \tau \\ 0, & x \in \sigma - \tau \end{cases} \tag{3.2}$$

Next we define the auxiliary function

$$v(x) = \tilde{u}(x) - w(x), \qquad x \in \sigma, \tag{3.3}$$

where

$$w(x) = \frac{1}{4\pi}\int_\sigma \frac{k^2\tilde{u}(x')}{|x-x'|}\,dV' = \frac{1}{4\pi}\int_\tau \frac{k^2u(x')}{|x-x'|}\,dV' \tag{3.4}$$

We see that w(x) is simply the potential due to the volume density $-k^2u(x)$ which is certainly in $C^{(\lambda)}(\overset{\circ}{\tau})$, since u is analytic in $\overset{\circ}{\tau}$. Thus we have for $x \in \overset{\circ}{\tau}$ (Kellogg [10] p. 156)

$$\Delta w = -k^2u(x). \tag{3.5}$$

Hence v is harmonic in $\overset{\circ}{\tau}$, since for $x \in \overset{\circ}{\tau}$

$$\Delta v = \Delta u + k^2u = 0. \tag{3.6}$$

Next, upon applying Lemma 3.1 to w(x) in σ, we see that $w \in C^{(1+\lambda)}(\sigma)$. Using the fact that F_1 is a $C^{(2+\lambda)}$ surface element, it is easy to show that we have $w \in C_S^{(1+\lambda)}(\overset{\circ}{F}_1)$. Since u = 0 on $\overset{\circ}{F}$, we see from (3.3) that $v \in C_S^{(1+\lambda)}(\overset{\circ}{F}_1)$. Upon applying Theorem 3.1 to the harmonic function v in the region τ,(i.e. τ_o and F_1 are

respectively the G and F of Theorem 3.1), we find that $v \in C^{(1+\lambda)}(\tau)$. Hence $u \in C^{(1+\lambda)}(\tau)$. From this and (3.2) it follows that $\tilde{u} \in C^{(\lambda)}(\sigma)$. The second part of Lemma 3.1 now implies that $w \in C^{(2+\lambda)}(\sigma)$; therefore $w \in C_S^{(2+\lambda)}(\overset{\circ}{F}_1)$, and consequently $v \in C_S^{(2+\lambda)}(\overset{\circ}{F}_1)$. Again applying Theorem 3.1 to v in τ, we get $v \in C^{(2+\lambda)}(\tau)$, and hence $u \in C^{(2+\lambda)}(\tau)$.

To complete the proof of the theorem, we choose for every interior point $x \in G$ a number $\rho_1(x) > 0$ such that $\sigma(\rho_1,x) \subset G$. Any compact subset A of $G \cup \overset{\circ}{F}$ is included in the union of a finite number of spheres $\sigma(\rho_1(x_i)/2,x_i)$ and $\sigma(\rho_o(\hat{x}_j)/2,\hat{x}_j)$, where the x_i, $i = 1,2,\ldots, n$, are interior points of G, and the $\hat{x}_j$, $j = 1,2,\ldots,m$, belong to $\overset{\circ}{F}$. Since u is analytic in $\sigma(\rho_1(x_i),x_i)$, and $u \in C^{(2+\lambda)}(\tau_{G\cup\overset{\circ}{F}}(\rho_o(\hat{x}_j),\hat{x}_j))$, it is easy to see that every D^2u satisfies a uniform Holder condition with exponent λ on A. But A is an arbitrary compact subset of $G\cup\overset{\circ}{F}$, whence the theorem follows.

Smoothness at inner points of $F \in \mathcal{H}$ will be established by applying certain existence and uniqueness theorems proved by Miranda [14]; these deal with the mixed boundary value problem for a linear second order elliptic equation in a finite domain. The following notation is useful in connection with these results, as well as the statements and proofs of the later theorems of this section.

For any point x and any subset A of E^n, $d(x,A)$ will denote the distance $\inf_{y \in A} |x-y|$ of x from A.

We shall denote functions of x by small letters, e.g. $u(x)$, $p(x)$, $q(x)$, ...; the corresponding capital letters U, P, Q, ..., will be used, with suitable subscripts, to denote certain functional quantities as follows: if for some region R, $u \in C^{(n)}(R)$, then for $0 \leqq m \leqq n$ we define

$$U_m[R] = \sum \sup_{x \in R} |D^m u(x)|; \tag{3.7}$$

if $u \in C^{(m+\lambda)}(R)$, we define

$$U_{m+\lambda}[R] = \sum \sup_{x,y \in R} \frac{|D^m u(x) - D^m u(y)|}{|x-y|^\lambda} . \tag{3.8}$$

The sums are to be taken over all derivatives of order m. If R is not closed, the quantities $U_m[R]$, $U_{m+\lambda}[R]$ may, of course, be $+\infty$. Similarly, if $\phi \in C_S^{(n)}(F)$, $0 \leq n \leq 2$, for some $C^{(2+\lambda_1)}$ surface element F with the representation (2.2), then for $0 \leq m \leq n$ we define*

$$\Phi_m^S[F] = \sum_{\substack{k_1+k_2=m \\ k_1,k_2 \geq 0}} \sup_{(x_1,x_2)\in\bar{D}} |D_1^{k_1} D_2^{k_2} \phi(x_1,x_2,f(x_1,x_2)| ; \tag{3.9}$$

if $\phi \in C_S^{(m+\lambda)}(F)$, where $m+\lambda \leq 2+\lambda_1$, $0 < \lambda < 1$, we define

$$\Phi_{m+\lambda}^S[F] = \sum_{\substack{k_1+k_2=m \\ k_1,k_2 \geq 0}} \sup_{\substack{(x_1,x_2)\in\bar{D} \\ (y_1,y_2)\in\bar{D}}} \frac{|D_1^{k_1} D_2^{k_2} \phi(x_1,x_2,f(x_1,x_2)-D_1^{k_1} D_2^{k_2} \phi(y_1,y_2,f(y_1,y_2)|}{[(x_1-y_1)^2 + (x_2-y_2)^2]^{\lambda/2}} . \tag{3.10}$$

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* $D_1\phi$ and $D_2\phi$ denote the derivatives of ϕ with respect to x_1 and x_2. It should be noted that the quantities defined in (3.9) and (3.10) depend in an essential way upon the particular coordinate system x_1, x_2, x_3 used to represent F. However, this does not cause any difficulty; in each instance where these quantities occur, we shall assume that some suitable system of coordinates has been affixed to F throughout the discussion.

The existence and uniqueness theorem which we shall use to prove the required smoothness at $\overset{\circ}{F}$ for $F \in \mathcal{H}$ comprises a somewhat weakened form of Miranda's results. It deals with a differential operator L in a finite domain H described in the following paragraphs.

(a) H is the interior of a regular closed surface lying in the region $x_3 \geq 0$. $\dot{H}$ is divided into two parts B_D and B_N, each consisting of the union of a finite number of $C^{(2+\lambda)}$ surface elements; we shall denote by F_j^D, $j=1,2,\ldots,m_D$, and F_i^N, $i=1,2,\ldots,m_N$, the elements forming B_D and B_N respectively. Every point of B_D, (B_N), has a neighborhood whose intersection with B_D, (B_N), is contained in F_i^D, (F_j^N), for at least one $i,(j)$. (Unlike the set of surface elements in hypothesis (c) of Theorem 2.1, two elements F_1^D and F_2^D (or F_1^N and F_2^N) may have more than edge points in common.)

(b) Let us denote the sets $\bigcup_{j=1}^{m_D} \overset{\circ}{F}{}_j^D$ and $\bigcup_{i=1}^{m_N} \overset{\circ}{F}{}_i^N$ by $\overset{\circ}{B}_D$ and $\overset{\circ}{B}_N$ respectively; the points of these sets will be called <u>inner points</u> of B_D and B_N. Also, let $\dot{B}_D = B_D - \overset{\circ}{B}_D$ and $\dot{B}_N = B_N - \overset{\circ}{B}_N$. Then B_D has no points in common with the plane $x_3 = 0$ other than the points of $J = \dot{B}_D \cap \dot{B}_N = \dot{B}_D = \dot{B}_N$.

(c) For $x \in J$, let $\alpha(x)$ be the angle between the positive x_3-axis and the outgoing normal* to B_D at x; then there exists an $\alpha_o > 0$ such that $\alpha_o \leqq \alpha(x) \leqq \pi - \alpha_o$.

(d) L is a second order linear elliptic differential operator with real coefficients defined in $\bar{H}$. It is given by

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* If neither normal to B_D at x is directed into the interior of H, i.e. if the "interior angle" between B_D and B_N is $\leqq \pi/2$, the outgoing normal is the one which points into the region $x_3 > 0$, or, in case the normals lie in the plane $x_3 = 0$, the one directed outward from B_N. Thus it is the limiting position at x of the outgoing normal at neighboring inner points of B_D.

$$Lu = \sum_{i,j=1}^{3} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{3} p_i(x) \frac{\partial u}{\partial x_i} \tag{3.11}$$

where $a_{ij} = a_{ji}$.

(e) $a_{ij}(x) \in C^{(1+\lambda)}(\bar{H})$; $p_i(x) \in C^{(\lambda)}(\bar{H})$. Thus there must exist a positive number K such that for $0 \leqq i, j \leqq 3$, we have, suppressing the subscripts of the a_{ij}, and p_i,

$$A_o[\bar{H}],\ A_\lambda[\bar{H}],\ P_o[\bar{H}],\ P_\lambda[\bar{H}] \leq K \tag{3.12}$$

(f) There exists a number $\Lambda > 0$ such that for any real vector ξ, and any $x \in \bar{H}$,

$$\Lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^{3} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \tag{3.13}$$

The boundary value problem we are concerned with is given by

$$Lu = q(x), \qquad x \in \bar{H} \tag{3.14}$$

$$u = \phi(x), \qquad x \in B_D \tag{3.15}$$

$$\sum_{i=1}^{3} a_{i3}(x) \frac{\partial u}{\partial x_i} + \beta(x)u = \psi(x), \qquad x \in \overset{o}{B}_N \tag{3.16}$$

Here the functions q, β, ψ; and ϕ are all real and satisfy the following conditions.

(g) $q \in C^{(\lambda)}(\bar{H})$

(h) $\beta \geqq 0$, and for $1 \leqq j \leqq m_N$, $\beta, \psi \in C_S^{(1+\lambda)}(F_j^N)$. Thus for β, in particular, there exists a $K_1 > 0$ such that

$$\sum_{j=1}^{m_N} \left(B_o^S[F_j^N] + B_{1+\lambda}^S[F_j^N]\right) \leqq K_1 \tag{3.17}$$

(i) For $1 \leqq i \leqq m_D$, $\phi \in C_S^{(\lambda)}(F_i^D) \cap C_S^{(2+\lambda)}(\overset{o}{F}{}_i^D)$. Moreover, denoting $d(x,J)$ by δ_x, we have, for $1 \leqq i \leqq m_D$,

$$\sup_{x \in \overset{o}{F}{}_i^D} \left\{ \delta_x^2 \, \Phi_{2+\lambda}^S \left[F_i^D \cap \sigma(\tfrac{3}{4}\delta_x, x)\right] \right\} < \infty \tag{3.18}$$

<u>Theorem 3.3</u>. The mixed boundary value problem (3.14), (3.15), (3.16), where the region H, boundary surfaces B_D and B_N, operator L, and functions q, β, ϕ and ψ satisfy conditions (a) through (i), has one and only one solution u in the class $C^{(0)}(\bar{H}) \cap C^{(1)}(\bar{H}-B_D) \cap C^{(2)}(H)$. Moreover, for some $\tilde{\lambda}$, $0 < \tilde{\lambda} \leqq \lambda$, depending only on λ, α_o, and Λ,

$$u \in C^{(2+\tilde{\lambda})}(\bar{H}-J) \tag{3.19}$$

The existence of a solution satisfying (3.19) is asserted by Theorem 6, I of [14]. The uniqueness in the indicated class of functions is proved in footnote (13) of [14]. These results are proved by Miranda for equations in n variables with less stringent conditions on L and the functions q, ψ and β.

Next we need the existence of a $C^{(2+\lambda)}$ coordinate transformation of the whole space which "flattens" a given $C^{(2+\lambda)}$ surface element, and possesses certain other suitable properties. This is provided by

Lemma 3.2 For a given $C^{(2+\lambda)}$ surface element F there exists a 1-1 coordinate transformation $x_i' = g_i(x)$, $i = 1,2,3$, or, more concisely, $x' = g(x)$, with the following properties:

(a) in the new coordinate system, F is represented by a surface element F' lying in the plane $x_3' = 0$;

(b) the functions $g_i(x)$ as well the functions comprising the inverse transformation belong to $C^{(2+\lambda)}(E^3)$;

(c) for all x, $$\frac{\partial(g_1,g_2,g_3)}{\partial(x_1,x_2,x_3)} \equiv 1$$

(d) the Laplacian operator is transformed into an operator L in the x' coordinates which satisfies conditions (d), (e), and (f) of Theorem 3.3 in any region R', the constants K_o and Λ being independent of R';

(e) there exists a constant C_1 such that*

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* If a and b are the number triples (a_1,a_2,a_3) and (b_1,b_2,b_3) respectively, $|a-b|$ denotes the quantity $\left[\sum_{i=1}^{3}(a_i-b_i)^2\right]^{1/2}$ no matter what the number triples a and b represent. Similarly, if A is a set of number triples, $d(a,A) = \inf_{b \in A} |a-b|$ regardless of how the number triples are interpreted.

$$C_1^{-1}|x-y| \leq |x'-y'| \leq C_1|x-y| \; ; \tag{3.20}$$

(f) let us denote differentiation with respect to the x' coordinates by D'u, and let us attach a prime to the quantities defined in (3.7) and (3.8) to indicate that the derivatives involved are with respect to the x' coordinates; then there exists a constant C_2 such that for any complex function u at any point where ∇u is defined we have

$$C_2^{-1}|Du| \leq |D'u| \leq C_2|Du| \; ; \tag{3.21}$$

if $U_1[R] < \infty$ for some region R, then

$$C_2^{-1}U_1[R] \leq U_1'[R'] \leq C_2U_1[R] \; ; \tag{3.22}$$

here R' is the region corresponding to R in the space of triples x';

(g) if in some region R, $u \in C^{(2+\lambda)}(R)$, then $u \in C^{(2+\lambda)}(R')$ (where u is regarded as a function of the x' coordinates), and conversely; moreover, there exists a constant C_3, independent of u or R, such that if $U_{2+\lambda}[R] < \infty$, we have

$$U_{2+\lambda}'[R'] \leq C_3(U_{2+\lambda}[R] + U_2[R] + U_1[R]) \; ; \tag{3.23}$$

(h) suppose that $\phi(x)$ is a function defined on F, and in some coordinate system in which F has the representation (2.2) we set $\phi(x_1,x_2,f(x_1,x_2) = \tilde{\phi}(x_1,x_2)$; then we can choose the transformation g(x) so that ϕ is given on F' by the same function $\tilde{\phi}(x_1',x_2')$.

<u>Proof</u>: Let F be represented by (2.2), the coordinate system being the one with respect to which $\tilde{\phi}$ is given, if it is to be preserved as stated in (h).

According to Definition 3 (Section 2), there exists a domain $\tilde{D}$ containing $\bar{D}$, and a function $\tilde{f}(x_1,x_2)$ in $C^{(2+\lambda)}(\tilde{D})$ which is equal to $f(x_1,x_2)$ on $\bar{D}$. Now let D_1 be a domain such that $\bar{D} \subset D_1$, and $\bar{D}_1 \subset \tilde{D}$. Let $\omega(x_1,x_2)$ be a function defined over the entire x_1-x_2 plane which is equal to 1 on $\bar{D}$ and 0 in the complement of D_1, and belongs to $C^{(3)}(E^2)^*$. We now construct a function in $C^{2+\lambda}(E^2)$ which is equal to $f(x_1,x_2)$ in $\bar{D}$, namely

$$\eta(x_1,x_2) = \begin{cases} \tilde{f}(x_1,x_2)\omega(x_1,x_2) , & (x_1,x_2) \in D_1 \\ 0 & (x_1,x_2) \notin D_1 . \end{cases} \tag{3.24}$$

Then the transformation and its inverse, given respectively by

$$\begin{cases} x_1' = x_1 , \\ x_2' = x_2 , \\ x_3' = x_3 - \eta(x_1,x_2) , \end{cases} \quad \text{and} \quad \begin{cases} x_1 = x_1' , \\ x_2 = x_2' , \\ x_3 = x_3' + \eta(x_1',x_2') , \end{cases} \tag{3.25}$$

possess the properties (a), (b), (c) and (h). To show (d) we transform the Laplacian using (3.25); we get an operator L of the form (3.11) in the x' coordinates which is easily seen to satisfy conditions (d) and (e) of Theorem 3.3. The matrix $[a_{ij}]$ is given by

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* It is easy to show that such an ω exists. In fact, there are functions in $C^{(\infty)}(E^2)$ satisfying the given conditions; see [22], for example.

$$\begin{bmatrix} 1 & 0 & -\eta_{x_1'} \\ 0 & 1 & -\eta_{x_2'} \\ -\eta_{x_1'} & -\eta_{x_2'} & 1+\eta^2_{x_1'}+\eta^2_{x_1'} \end{bmatrix}$$

For any vector $\xi = (\xi_1, \xi_2, \xi_3)$, we have

$$\sum_{i,j=1}^{3} a_{ij}\xi_i\xi_j = \xi_3^2 + (\eta_{x_1'}\xi_3-\xi_1)^2 + (\eta_{x_2'}\xi_3-\xi_2)^2 \tag{3.26}$$

Now let $M \geqq 1$ be such that for $(x_1,x_2) \in E^2$

$$\left|\eta_{x_1'}\right| , \left|\eta_{x_2'}\right| \leqq M . \tag{3.27}$$

Then it is easy to show that for all real vectors ξ, (3.13) holds with $\Lambda = 12M^2$, whence (d) follows. The proofs of the remaining properties, (e), (f), and (g) present no difficulty, and will be omitted.

It is convenient to introduce the notation ζ for any hemisphere $(x_1-a)^2+(x_2-b)^2+x_3^2 < \rho^2$, $x_3 \geqq 0$; γ will denote the open disk comprising the flat part of $\dot{\zeta}$, and Z will denote the curved part of $\dot{\zeta}$, i.e. $\dot{\zeta}-\gamma$. It is clear that $\overset{\circ}{\zeta}$ satisfies the conditions (a), (b) and (c) of Theorem 3.3 on the domain H, $\bar{\gamma}$ and Z corresponding to B_N and B_D respectively.

We can now prove the analogue of Theorem 3.2 for surface elements F in $\mathcal{H}$, namely

<u>Theorem 3.4</u> Let G and F be as in Theorem 3.1, and let $\beta(x)$, defined on F, be

a non-negative real function of class $C_S^{(1+\lambda)}(F)$. Suppose that u in $C^{(0)}(G \cup F)$ satisfies the wave equation (2.4) in G, and that for $\hat{x} \in \overset{\circ}{F}$, $\lim\limits_{\substack{x \to \hat{x} \\ x \in G}} \nabla u$ exists and u satisfies

$$\frac{\partial u}{\partial n} + \beta(\hat{x})u = 0. \tag{3.28}$$

Then for some $\tilde{\lambda}$, $0 < \tilde{\lambda} < 1$, $u \in C^{(2+\tilde{\lambda})}(G \cup \overset{\circ}{F})$.

Proof: By Lemma 3.2 there exists a transformation of coordinates $x' = g(x)$ which flattens F, the corresponding element F' lying in the plane $x_3' = 0$; also, assuming the representation (2.2) for F, g can be chosen so as to transform $\tilde{\beta}(x_1,x_2) = \beta(x_1,x_2,f(x_1,x_2))$ into the identical function $\tilde{\beta}(x_1',x_2')$ in $C_S^{(1+\lambda)}(F')$. Finally, by means of a reflection in the plane $x_3' = 0$, if necessary, we can see to it that the domain G', corresponding to G, lies on the side $x_3 > 0$ of F'.

Next, for any point $\hat{x}'$ of F', let $\hat{\delta} = \frac{1}{2} d(\hat{x}', \dot{G}'-F')$; by the hypothesis on F and G, $\hat{\delta} > 0$. Let ζ be the hemisphere of the type described above, with radius $\hat{\delta}$ and with $\hat{x}'$ as the center of its base γ, so that $\bar{\zeta} \subset G' \cup \overset{\circ}{F}'$, and $\bar{\gamma} \subset \overset{\circ}{F}'$. We shall show that for some $\tilde{\lambda}$, $0 < \tilde{\lambda} \leq \lambda$, where $\tilde{\lambda}$ is independent of $\hat{x}'$,

$$u(x') \in C^{(2+\tilde{\lambda})}(\zeta) \tag{3.29}$$

It will follow by (g) of Lemma 3.2 that $u(x) \in C^{(2+\lambda)}(g^{-1}(\zeta))$; then, since $g^{-1}(\zeta)$ includes the region $\tau_{G \cup \overset{\circ}{F}}(\rho,\hat{x})$ for sufficiently small ρ, (where

$\hat{x} = g^{-1}(\hat{x}'))$, the reasoning used at the end of the proof of Theorem 3.2 will yield the present theorem.

To prove (3.29), we begin by noting that, by (d) of Lemma 3.2, $u(x')$ is a solution of

$$Lu + k^2 u = 0, \qquad x' \in \mathfrak{g}\ , \tag{3.30}$$

where L satisfies conditions (d), (e) and (f) of Theorem 3.3; moreover, the number Λ in condition (f) is independent of $\hat{x}'$. The boundary condition (3.28) is transformed into (3.16)*. Next, let us set

$$\begin{aligned} v &= \mathrm{Re}\ \{u\} & w &= \mathrm{Im}\ \{u\} \\ q_1 &= \mathrm{Re}\ \{-k^2 u\} & q_2 &= \mathrm{Im}\ \{-k^2 u\} \end{aligned} \tag{3.31}$$

It is also convenient to assign a special notation for v and w on the surface Z, namely,

$$\phi(x') = v \qquad \chi(x') = w\ , \qquad x' \in Z\ . \tag{3.32}$$

Then v is a solution of the mixed boundary value problem, (in which only real functions are involved),

$$Lv = q_1(x')\ , \qquad x' \in \mathfrak{g}\ , \tag{3.33}$$

$$\sum_{i=1}^{3} a_{i3} \frac{\partial v}{\partial x_i'} + \beta v = 0\ , \qquad x' \in \gamma\ , \tag{3.34}$$

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* $\partial u/\partial n$ is transformed into the "conormal derivative" $\sum_{i=1} a_{i3} \frac{\partial u}{\partial x'}$

$$v = \emptyset(x') , \qquad\qquad x' \in Z . \tag{3.35}$$

w satisfies the same equations with q_1 and $\emptyset$ replaced by q_2 and X respectively. To these boundary value problems we wish to apply Theorem 3.3, of which all the conditions except (g) and (i) have already been established. We need discuss only the function v, identical considerations holding for w.

Condition (g) is easily shown to hold: as a result of the analyticity of u in G and the hypothesis concerning ∇u near $\overset{\circ}{F}$, we have

$$u(x) \in C^{(1)}(G \cup \overset{\circ}{F}) ; \tag{3.36}$$

then, by (f) of Lemma 3.2, $u(x') \in C^{(1)}(G' \cup \overset{\circ}{F}')$, whence q_1 is certainly in $C^{(\lambda)}(\bar{\zeta})$.

The first part of condition (i) of Theorem 3.3 follows from the fact that $u \in C^{(2+\lambda)}(G)$, (u being analytic in G), and hence, by Lemma 3.2 (g), $u(x') \in C^{(2+\lambda)}(G')$; it is easy to show from this that $u(x')$, and hence $v(x')$, or, by (3.32), $\emptyset(x')$, belongs to $C_S^{(2+\lambda)}(\overset{\circ}{F}{}_i^D)$ for any of the $C^{(2+\lambda)}$ surface elements F_i^D comprising Z (cf. condition (b) of Theorem 3.3). It remains to establish (3.18).

Let ζ_1 be the hemisphere in $G' \cup \overset{\circ}{F}'$ of radius $\cdot\frac{3}{2}\hat{\delta}$, with the center of its base at $\hat{x}'$, and let $I = g^{-1}(\bar{\zeta}_1)$; let us denote $d(x, \dot{I})$ by d_x. Now, for any solution u of (2.4) which is in $C^{(2+\lambda)}(\overset{\circ}{I})$ and bounded in I, the mean value theorem for the reduced wave equation implies*

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* Such "interior estimates" for solutions are proved in [15] for general elliptic systems of partial differential equations; in our special case, (3.37) can easily be proved using the mean value theorem ([24], p. 260). Cf. footnote to (3.45).

$$d_x^{\,2}\ |D^2u| \leq CU_0[I]\ , \qquad x \in \overset{\circ}{I} \tag{3.37}$$

Here C is a constant depending only on k and I. Since in our case Du is itself a solution of (2.4) which is bounded in I and analytic in $\overset{\circ}{I}$, we can apply (3.37) to Du in I and obtain

$$d_x^{\,2}\ |D^3u| \leq CU_1[I]\ , \qquad x \in \overset{\circ}{I}\ . \tag{3.38}$$

Let δ_x' denote the distance $d(x',J)$, where J has the same meaning as in condition (b) of Theorem 3.3, i.e. $J = Z \cap \bar{\gamma}$. Let $x' \in Z$ satisfy the condition

$$\delta_{x'} < \hat{\delta}/2\ . \tag{3.39}$$

Then every point of the sphere $\sigma = \sigma(\frac{3}{4}\,\delta_{x'},x')$ is at a distance $\geq \hat{\delta}/8$ from $\zeta_1 - \overset{\circ}{F}'$. Moreover,

$$\delta_{x'} < 8 \inf_{y'\epsilon\sigma}\ d(y',\dot{\zeta}_1) \tag{3.40}$$

Now, let x_μ be the point in $g^{-1}(\sigma)$ at which $|D^3u|$ is a maximum. Then, by (3.38) we have

$$d_{x_\mu}^{\,2}\ U_3[g^{-1}(\sigma)] \leq C_4 \tag{3.41}$$

where $C_4 = CU_1[I]$. Next, the convexity of ζ_1 and Lemma 3.2, (e), imply that there is a constant C_5, depending only on g(x) and the diameter of ζ_1, such that, for all λ_1 where $0 \leq \lambda_1 < 1$, we have

$$U_{2+\lambda_1}\left[g^{-1}(\sigma)\right] \leq C_5 U_3\left[g^{-1}(\sigma)\right] \tag{3.42}$$

Then by (3.23) of Lemma 3.2(g), (3.42) with $\lambda_1 = \lambda$ and $\lambda_1 = 0$, (3.36), (3.40), and (3.41), there exists a constant C_6 such that for all x' satisfying (3.39),

$$\delta_{x'}^{2}\, U'_{2+\lambda}\left[\sigma\left(\tfrac{3}{4}\,\delta_{x'}, x'\right)\right] \leq C_6 \,. \tag{3.43}$$

But this obviously implies (3.18), since by (3.32) and (3.31), $\phi = \mathrm{Re}\,\{u\}$ on Z.

We can now apply Theorem 3.3 to the boundary value problem (3.33), (3.34), and (3.35). Since $v(x') \in C^{(0)}(\bar{\zeta}) \cap C^{(1)}(\zeta) \cap C^{(2)}(\mathring{\zeta})$, it is the only solution of the mixed boundary value problem in that class, and therefore there exists a $\tilde{\lambda}$, where $0 < \tilde{\lambda} \leq \lambda$ such that

$$v(x') \in C^{(2+\tilde{\lambda})}(\zeta) \tag{3.44}$$

Moreover, since $\alpha_o \equiv \pi/2$ for any region ζ, and λ and Λ are independent of $\hat{x}'$ in F', $\tilde{\lambda}$ is independent of $\hat{x}'$. Since the identical conclusions hold for w, it follows that $u(x')$ satisfies (3.29), which completes the proof.

In the next section we shall need certain estimates for $|Du(x)|$ at an arbitrary point x of G, where u is a solution of (2.4); these estimates are in terms of $\sup|u|$ in a suitable neighborhood of x. Suppose ρ is such that the sphere $\sigma(\rho,x)$ lies entirely in G; upon applying the mean value theorem for the wave equation to Du, we get, for $\rho \leq 1$ and for any constant u_o,

$$\rho|Du(x)| \leq C_7 \sup_{y\epsilon\sigma(\rho,x)} |u(y)-u_o| \tag{3.45}$$ *

Here C_7 depends only on the propagation constant k. The remainder of this section is concerned with establishing estimates of this type when the sphere intersects one of the $C^{(2+\lambda)}$ surface elements F of $B = \dot{G}$. We shall obtain generalizations of (3.45) in which $\sigma(\rho,x)$ is replaced by $\tau_G(\rho,x)$. In particular we shall see that for homogeneous boundary conditions (3.45) need not be otherwise changed (in this case, however, the constant depends on the surface element F as well as on k); i.e. the boundary element F does not "get in the way". It is here that we make explicit use of the Schauder estimates for the derivatives of solutions of second order elliptic equations near a smooth boundary**. These are presented, for the cases in which we are interested, in the following two lemmas.

Lemma 3.3 Let ζ be a hemisphere*** of radius $\rho \leq 1$, and let L be a differential operator satisfying conditions (d), (e) and (f) of Theorem 3.3 with $\bar{\zeta}$ as the region $\bar{H}$. Suppose u in $C^{(2+\lambda)}(\zeta)$ is a solution of

$$Lu + k^2u = 0 , \qquad x \in \mathring{\zeta} \tag{3.46}$$

$$u = 0 \qquad x \in \gamma \tag{3.47}$$

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*Applying the mean value theorem to the wave function $\partial u/\partial x_3$, say, we get

(I) $\qquad 4\pi r \sin kr\, u_{x_3}(x)/k = \int_{\Sigma(r,x)} u_{x_3}\, dS .$

Let ρ_o be such that $\sin kr \geq r/2$ for $r \leq \rho_o$. For $\rho \leq \rho_o$, integrate (I) with respect to r from 0 to ρ; then apply Gauss's theorem to the resulting volume integral on the right hand side. The latter becomes

(II) $\qquad \int_{\Sigma(\rho,x)} u\, \partial x_3/\partial n\, dS = \int_{\Sigma(\rho,x)} (u - u_o)\, \partial x_3/\partial n\, dS .$

(3.45) then follows easily for all $\rho \leq 1$, for a suitable C_7.

**These estimates are in fact used in the proof of Miranda's existence theorem for the mixed boundary value problem, from which Theorem 3.3 is obtained.

***Here we are using the notation introduced just before Theorem 3.3.

Then we have for $x \in \zeta$,

$$d(x,Z)|Du(x)| \leq C_8 U_o[\zeta] \ ,$$

where C_8 depends on K, Λ, and $|k|$.

Lemma 3.4 Let ζ and L be as in Lemma 3.3, let q_o be an arbitrary complex constant, and let $\beta(x)$ and $\psi(x)$ be functions defined on $\bar{\gamma}$ which satisfy condition (h) of Theorem 3.3 with $\bar{\gamma}$ as the surface element $F_1^N = B_N$. Suppose u in $C^{(2+\lambda)}(\zeta)$ is a solution of

$$Lu + k^2 u = q_o, \qquad x \in \mathring{\zeta} \tag{3.49}$$

$$\sum_{i=1}^{3} a_{i3} \frac{\partial u}{\partial x_i} + \beta u = \psi, \qquad x \in \gamma \tag{3.50}$$

Then we have for $x \in \zeta$,

$$d(x,Z)|Du(x)| \leq C_9 \left(U_o[\zeta] + \rho^2 |q_o| + \rho^{2+\lambda} \Psi^S_{1+\lambda}[\bar{\gamma}] + \rho^2 \Psi^S_1[\bar{\gamma}] + \rho \Psi^S_o[\bar{\gamma}] \right) \tag{3.51}$$

where C_9 depends only on K, K_1, Λ, and $|k|$.

These lemmas are special cases (with unnecessarily strong hypothesis on L and the functions β and ψ) of the comprehensive Theorem 7.2 of [16]*.

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* This theorem covers general elliptic systems of equations with general boundary conditions, and also provides estimates for higher order derivatives of u. Simple proofs of the required estimates for a single second order equation may be found in [15,20] and [14] for the Dirichlet boundary condition and boundary condition (3.50) respectively; these proofs are restricted to equations with real coefficients and real u, ψ, etc., but only inessential changes are required to extend the proof to cover (3.46) and (3.49) with complex k^2, and with u and ψ complex.

<u>Theorem 3.5</u> Let G, F, and u be as in Theorem 3.2. Then for all x_o in G and $\rho \leqq 1$ such that $\tau_{\bar{G}}(\rho, x_o) \cap \dot{G} \subset F$, we have

$$\rho |Du(x_o)| \leqq C_{10} U_o [\tau_{\bar{G}}(\rho, x_o)] \tag{3.52}$$

where C_{10} depends only on $|k|$ and F.

<u>Proof:</u> Let $x' = g(x)$ be the coordinate transformation used in the proof of Theorem 3.4, the surface element F' lying in the plane $x_3' = 0$, and the region G' lying on the side $x_3' > 0$ of F'. (As usual, we shall attach primes to letters designating points and sets in order to represent corresponding points, i.e. number triples, and sets in the new coordinate system.) For this transformation g let us define

$$\theta = (1+2C_1{}^2)^{-1} \tag{3.53}$$

where C_1 is the constant of Lemma 3.2(e); note that, by (3.20), $C_1 \geqq 1$.

Now, suppose x_o and ρ satisfy the conditions stated in the theorem. We can restrict the proof to the case where

$$d(x_o, F) \leqq \theta\rho \tag{3.54}$$

and where

$$\rho \leqq (2C_1)^{-1} \tag{3.55}$$

For, if (3.54) does not hold, we can apply (3.45) to the sphere $\sigma(\theta\rho, x_o)$ which lies in G, setting $u_o = 0$, and obtain

$$\theta\rho|Du(x_o)| \leq C_7 U_o[\sigma(\theta\rho, x_o)] \leq C_7 U_o[\tau_{\bar{G}}(\rho, x_o)]; \tag{3.56}$$

thus (3.52) holds if we take $C_{10} = C_7/\theta$. Next, if (3.52) has been proved under the assumption that (3.55) is satisfied, the theorem follows easily for the case where $(2C_1)^{-1} < \rho \leq 1$; for upon substituting $(2C_1)^{-1}$ for ρ in (3.52) and noting that $U_o\left[\tau_{\bar{G}}\left((2C_1)^{-1}, x_o\right)\right] \leq U_o[\tau_{\bar{G}}(\rho, x_o)]$, we see that (3.52) holds if we replace C_{10} by $2C_1C_{10}$.

Assuming therefore that (3.54) and (3.55) hold, let x_1 be a point of $F \cap \tau_{\bar{G}}(\rho, x)$ such that

$$|x_o - x_1| \leq \theta\rho \tag{3.57}$$

Let T be the spherical part of the boundary of $\tau_{\bar{G}}(\rho, x_o)$, i.e. $T = \dot{\tau}_{\bar{G}}(\rho, x) \cap \sum(\rho, x_o)$. Then

$$d(x_1, T) \geq \rho - |x_o - x_1| \geq \rho(1-\theta). \tag{3.58}$$

By Lemma 3.2(e), (3.57), (3.58) and (3.53)

$$|x_o' - x_1'| \leq \frac{C_1^2\theta}{1-\theta}\, d(x_1', T') \leq \frac{1}{2}\, d(x_1', T') \,. \tag{3.59}$$

Now let ζ be the hemisphere contained in $g(\tau_{\bar{G}}(\rho, x))$ of radius $\frac{3}{4}d(x_1', T')$, with its base γ on F', the center of γ being x_1'. By (3.55) and Lemma 3.2(e), the radius of ζ is < 1. By Lemma 3.2(d), equation (2.4) is transformed into (3.46), where L satisfies the conditions of Lemma 3.3. The constants K and Λ associated with L are independent of x_1' or ζ, being determined once and for all by the transformation g and by $|k|$. By Theorem

3.2, $u \in C^{(2+\lambda)}(G \cup \overset{\circ}{F})$, and therefore by Lemma 3.2(g), $u(x') \in C^{(2+\lambda)}(G' \cup \overset{\circ}{F}')$. Consequently, we can apply Lemma 3.3 to u in ζ. We get

$$d(x_o', Z)\ |D'u(x_o')| \leq C_8 U_o[\zeta] \tag{3.60}$$

where C_8 depends only on g and $|k|$.

By (3.59), and Lemma 3.2(e), we have

$$d(x_o', Z) \geq \frac{1}{4}\, d(x',T') \geq \frac{1}{4}\, C_1^{-1}\rho \tag{3.61}$$

By (3.60), (3.61), and (3.21) we get

$$\rho|Du(x_o)| \leq 4\ C_1C_2C_8U_o[g^{-1}(\zeta)] \leq 4\ C_1C_2C_8U_o[\tau_{\bar{G}}(\rho,x_o)] \tag{3.62}$$

Since the constants depend only on g - hence on F - and on $|k|$ (in the case of C_8), this completes the proof.

We now give the analogous estimate near a $C^{(2+\lambda)}$ surface element on which u satisfies a homogeneous "impedance boundary condition".

<u>Theorem 3.6</u> Let G, F, β , and u be as in Theorem 3.4. Then for all x_o in G and $\rho \leq 1$, such that $\tau_{\bar{G}}(\rho,x_o) \cap \dot{G} \subset F$, and any complex constant u_o, we have

$$\begin{aligned}\rho|Du(x_o)| \leq\ & C_{11}\Big(\sup_{x\in\tau_{\bar{G}}(\rho,x_o)}|u-u_o| + \rho^2|k^2u_o| + \rho^{2+\tilde{\lambda}}|u_o|B^S_{1+\tilde{\lambda}}[F] \\ & + \rho^2|u_o|B^S_1[F] + \rho|u_o|B^S_o[F]\Big)\end{aligned} \tag{3.63}$$

Here C_{11} depends only on F, $|k|$ and $B_0^S[F] + B_{1+\tilde{\lambda}}^S[F]$; $\tilde{\lambda}$ depends only on F and λ, and $0 < \tilde{\lambda} \leq \lambda$.

Proof: Let $v = u - u_0$. Then v satisfies the equations

$$\Delta v + k^2 v = -k^2 u_0 \qquad x \in G \tag{3.64}$$

$$\frac{\partial v}{\partial n} + \beta(x)v = -u_0\beta(x) \qquad x \in \overset{\circ}{F} \tag{3.65}$$

By Theorem 3.4, we see that u, and therefore v, is in $C^{(2+\tilde{\lambda})}(G \cup \overset{\circ}{F})$ for some $\tilde{\lambda}$, $0 < \tilde{\lambda} \leq \lambda$, and hence $v(x') \in C^{(2+\tilde{\lambda})}(G' \cup \overset{\circ}{F}')$. From this point on the proof is essentially the same as that of Theorem 3.5. Here the coordinate transformation g is chosen, using Lemma 3.2(h), so as to leave $\tilde{\beta}(x_1,x_2) = \beta(x_1,x_2,f(x_1,x_2))$ unchanged, the representation (2.2) being assumed for F; the coordinate system x_1,x_2,x_3 is the one with respect to which the quantities $B_{1+\tilde{\lambda}}^S[F]$, etc. are defined. Hence $B_{1+\tilde{\lambda}}^S[F] = B_{1+\tilde{\lambda}}^S[F']$, $B_1^S[F] = B_1^S[F']$, etc. Finally, in place of Lemma 3.3, we apply Lemma 3.4 to $v(x')$ in ζ, with $\tilde{\lambda}$ in place of λ, $q_0 = -k^2 u_0$, and $\psi = -u_0\beta$.

4. On the behavior of solutions near edges

The purpose of this section is to prove

Theorem 4.1 Let G, B and u satisfy the hypothesis of Theorem 2.1, with the possible exception of the radiation condition, and let $\mathfrak{E}$ be the set of edge points of the surface elements in hypothesis (c). Then

$$Du(x) = o(1/d(x, \mathfrak{E})) , \qquad \text{as } d(x, \mathfrak{E}) \to 0 , \tag{4.1}$$

uniformly for $x \in G$.

The basic idea of the proof may be illustrated in the simple case of two half-planes with a common edge forming a right angle. Let the half-planes be given by $x_1=0$, $x_2\geqq 0$, and $x_2=0$, $x_1\geqq 0$, and let G be the region outside the first quadrant (with respect to the variables x_1,x_2); thus $\mathfrak{E}$ is the edge $x_1=x_2=0$. For $x\epsilon G$, let $\rho=d(x,\mathfrak{E})$. If x is in the third quadrant, the sphere $\sigma(\rho,x)$ lies in G; hence setting $u_o=u(x)$ in (3.45) we have

$$|Du(x)| \leq \frac{C_{10}}{\rho} \sup_{y\epsilon\sigma(\rho,x)} |u(y)-u(x)| \ . \tag{4.2}$$

If x is not in the third quadrant, $\sigma(\rho,x)$ no longer lies in G; however, if u satisfies either the boundary condition $u = 0$, or the condition $\frac{\partial u}{\partial n} = 0$ on each half plane, we can continue the solution through each half-plane by reflection, and thus obtain (4.2) with $\sigma(\rho,x)$ replaced by $\tau_G(\rho,x)$. Next, as $x \to \mathfrak{E}$, i.e. as $\rho \to 0$,

$$\sup_{y\epsilon\tau_G(\rho,x)} |u(y) - u(x)| = o(1) \ , \tag{4.3}$$

since u is continuous in $\bar{G}$. Moreover, the estimate (4.3) holds uniformly for x in any bounded subregion of G_1 of G, since u is uniformly continuous in any bounded subregion of $\bar{G}$. Hence we get (4.1) uniformly for $x\epsilon G_1$. (In this example, where the edge extends to infinity, we prove the uniformity of the estimate (4.1) only in bounded subregions.)

Theorems 3.5 and 3.6 enable us to deal in a similar manner with edges of $C^{(2+\lambda)}$ surface elements. However, in order to handle the general types of edge points which may occur under the hypotheses of Theorem 4.1, we need the following geometrical result. The proof is given in Appendix II.

Lemma 4.1 Let G be the exterior of a regular closed surface which is subdivded into a set $\mathcal{F}$ of $C^{(2+\lambda)}$ surface elements, and let $\mathcal{E}$ be the set of edge points of the elements of $\mathcal{F}$. Then there exists a number θ, where $0 < \theta \leq 1$, such that for $x \epsilon G$, the set $\tau_{\bar{G}}(\theta d(x, \mathcal{E}), x)$ intersects at most one element of $\mathcal{F}$.

Proof of Theorem 4.1: Let $\sigma(r,0)$ be a sphere containing B in its interior, and such that the distance from $\sum(r,0)$ to any point of B exceeds 2; let $G_r = G \cap \sigma(r,0)$. Suppose that $0 < \epsilon \leq 1$. By the uniform continuity of u in $\bar{G}_r$, there exists a $\delta > 0$ such that

$$|u(x) - u(y)| < \epsilon , \qquad x, y \in \bar{G}_r, \ |x-y| < 2\delta . \tag{4.4}$$

We can obviously assume that $\delta \leq \epsilon$.

Let θ be the constant of Lemma 4.1 for the domain G. For $d(x, \mathcal{E}) < \delta$ every point of the set $\tau = \tau_{\bar{G}}(\theta d(x, \mathcal{E}), x)$ is at a distance $< 2\delta$ from $\mathcal{E}$; moreover, τ is included in $\bar{G}_r$ and intersects at most one element $\hat{F}_i$ in $\mathcal{J}$ or F_j in $\mathcal{H}$. First, let us consider the case where $\tau \cap \hat{F}_i \neq \emptyset$ for some $\hat{F}_i \in \mathcal{J}$. By Theorem 3.5 there exists a constant $C_{10}^{(i)}$, depending only on $\hat{F}_i$ and $|k|$ such that

$$\theta d(x, \mathcal{E}) |Du(x)| \leq C_{10}^{(i)} U_o[\tau] < C_{10}^{(i)} \epsilon . \tag{4.5}$$

The second inequality follows from (4.4) and the fact that $u(y) = 0$ for some $y \epsilon \tau$. Next, suppose that $\tau \cap F_j \neq \emptyset$ for some $F_j \in \mathcal{H}$. Let $u_o = u(x)$ in Theorem 3.6, and note that $\theta d(x, \mathcal{E}) < \epsilon$. Then there exist constants $\tilde{\lambda}_j$, where $0 < \tilde{\lambda} \leq \lambda$, and $C_{11}^{(j)}$ such that

$$\theta d(x, \mathfrak{B})|Du(x)| \leq C_{11}^{(j)} \Big(\epsilon + \epsilon^2|k^2u(x)| + \epsilon^{2+\tilde{\lambda}}|u(x)|B^S_{1+\tilde{\lambda}_j}[F_j]$$

$$+ \epsilon^2|u(x)|B^S_1[F_j] + \epsilon|u(x)|B^S_o[F_j]\Big) \tag{4.6}$$

Here λ_j depends only on F_j and λ, and $C_{11}^{(j)}$ depends only on F_j, $|k|$. and $B^S_o[F_j] + B^S_{1+\tilde{\lambda}}[F_j]$. For $\tau \cap B = \emptyset$, we can still use (4.6) to estimate $Du(x)$, if $\mathcal{H}$ is not empty; if $\mathcal{H}$ is empty, then (4.5) still holds, since $u(\hat{x}) = 0$ for $\hat{x} \epsilon \mathfrak{B}$.

Since there are only a finite numbe1 of elements in $\mathcal{L} \cup \mathcal{H}$, there exists a constant M such that

$$C_{10}^{(i)} < M , \qquad \hat{F}_i \in \mathcal{L} \tag{4.7}$$

$$C_{11}^{(j)}\left[1 + U_o[\bar{G}_r] \left(|k|^2 + B^S_{1+\tilde{\lambda}_j}[F_j] + B^S_1[F_j] + B^S_o[F_j]\right)\right]$$

$$< M, \qquad F_j \in \mathcal{H}. \tag{4.8}$$

Then, since $\epsilon \leq 1$, we have

$$d(x, \mathfrak{B})|Du(x)| < M\theta^{-1}\epsilon, \tag{4.9}$$

for all x such that $d(x, \mathfrak{B}) < \delta$.

5. Proof of the uniqueness theorem for a regular closed surface.

Our proof of Theorem 2.1 is basically the same as that given by Wilcox[9]; as indicated in the introduction, Wilcox's proof was restricted to solutions u in $C^{(2)}(\bar{G})$, so that edges were implicitly ruled out. The difficulty with regard to singularities of u at edges is taken care of by the following lemma, whose proof, of course, depends essentially on Theorem 4.1. The lemma is a special case of Green's theorem for the regions $G_\rho = G \cap \sigma(\rho,0)$, where $B \subset \sigma(\rho,0)$.

Lemma 5.1 Let G, B, and u satisfy the hypothesis of Theorem 2.1 with the possible exception of the radiation condition. Then

$$\int_{G_\rho} \bar{u} \, \Delta u \, dV + \int_{G_\rho} \nabla \bar{u} \cdot \nabla u \, dV = \int_{\dot{G}_\rho} \bar{u} \frac{\partial u}{\partial n} \, dS \tag{5.1}$$

Proof: Suppose $\epsilon > 0$. By Theorem 4.1 there exists a $\delta_o > 0$ such that $d(x, \mathfrak{E})|Du| < \epsilon$ for $d(x, \mathfrak{E}) \leq \delta_o$. We can obviously assume that δ_o is less than the distance from B to $\sum(\rho,0)$, and also that $\delta_o \leq 1$. We now construct a lattice of cubes on E^3 of side $\delta \leq \delta_o/(2\sqrt{3})$. Let A_1 be the union of all (closed) cubes which intersect $\mathfrak{E}$, and let A_2 be the union of all cubes intersecting A_1. We note that for $x \in \dot{A}_2$,

$$\delta < d(x, \mathfrak{B}) \leq \delta_o \, . \tag{5.2}$$

We shall denote by ξ_δ that component of $G_\rho - A_2$ whose closure contains $\sum(\rho,0)$; the part of $\dot{\xi}_\delta$ which is included in $\dot{A}_2$ will be denoted by Ξ_δ.

It is easy to show that there are less than $8(L/\delta + 1)$ cubes in A_1, where L is the sum of the lengths of the edges of all F in $\mathscr{L} \cup \mathscr{H}$, and that consequently the number of cubes in A_2 is less than $27 \cdot 8(L/\delta + 1)$. Then the area of Ξ_δ is less than $6 \cdot 27 \cdot 8(L + 1)\delta$.

We now apply Green's first identity ([10], p 212) to the functions u and $\bar{u}$ in ξ_δ. This is valid since, by Theorems 3.2 and 3.4, $u \in C^{(2+\tilde{\lambda})}(\bar{\xi}_\delta)$. We have

$$\int_{\xi_\delta} \bar{u} \, \Delta u \, dV + \int_{\xi_\delta} \nabla \bar{u} \cdot \nabla u \, dV = \int_{\sum(\rho,0) \cup (B \cap \bar{\xi}_\delta)} \bar{u} \frac{\partial u}{\partial n} \, dS + \int_{\Xi_\delta} \bar{u} \frac{\partial u}{\partial n} \, dS \tag{5.3}$$

It follows from the choice of δ_o and (5.2) that for $x \in \Xi_\delta$, $|Du| < \epsilon/\delta$; we then have $\left|\frac{\partial u}{\partial n}\right| \leq |\nabla u| < \sqrt{3}\, \epsilon/\delta$. Hence

$$\left| \int_{\Xi_\delta} \bar{u} \frac{\partial u}{\partial n} \, dS \right| < 6 \cdot 27 \cdot 8 \sqrt{3}\, (L+1)\, U_o[G_\rho]\, \epsilon \tag{5.4}$$

Thus, as $\delta \to 0$, the second integral on the right of (5.3) approaches zero. The first integral on the left approaches $\int_{G_\rho} \bar{u} \Delta u \, dV$, which exists and is finite since $\bar{u}\Delta u = -k^2|u|^2$ in G_ρ. Similarly, the first integral on the right approaches

$$\int_{\sum(\rho,0)} \bar{u} \frac{\partial u}{\partial r} \, dS - \int_{\bigcup_{F_j \in \mathscr{H}} F_j} \beta_j |u|^2 dS = \int_{\dot{G}_\rho} \bar{u} \frac{\partial u}{\partial n} \, dS \tag{5.5}$$

where β_j is the "impedance" function on the element F_j. Consequently, the remaining integral approaches a finite limit, namely $\int_{G_\rho} \nabla u \cdot \nabla \bar{u} \, dV$, and we get (5.1)

Proof of Theorem 2.1: The radiation condition (2.5) may be written

$$\lim_{\rho\to\infty} \int_{\Sigma(\rho,0)} \left(\left|\frac{\partial u}{\partial r}\right|^2 + |k|^2 |u|^2 + i\bar{k}\bar{u} \frac{\partial u}{\partial r} - iku \frac{\partial \bar{u}}{\partial r} \right) dS = 0 \tag{5.6}$$

Upon substituting $-k^2 u$ for Δu in (5.1) and multiplying through by ik, we get

$$i\bar{k} \int_{\Sigma(\rho,0)} \bar{u} \frac{\partial u}{\partial r} dS = -i\bar{k} \int_B \bar{u} \frac{\partial u}{\partial n} dS - ik|k|^2 \int_{G_\rho} |u|^2 dV + i\bar{k} \int_{G_\rho} |\nabla u|^2 dV \tag{5.7}$$

Next we add the complex conjugate of (5.7) to (5.7), obtaining

$$\int_{\Sigma(\rho,0)} \left(i\bar{k}\bar{u} \frac{\partial u}{\partial r} - iku \frac{\partial \bar{u}}{\partial r} \right) dS = -2 \operatorname{Im} \left\{ k \int_B u \frac{\partial \bar{u}}{\partial n} dS \right\} + 2 \operatorname{Im} \{k\} |k|^2 \int_{G_\rho} |u|^2 dV + 2 \operatorname{Im} \{k\} \int_{G_\rho} |\nabla u|^2 dV \tag{5.8}$$

Upon substituting this into (5.6), we get, as $\rho \to \infty$,

$$\int_{\Sigma(\rho,0)} \left|\frac{\partial u}{\partial r}\right|^2 dS + |k|^2 \int_{\Sigma(\rho,0)} |u|^2 dS + 2 \operatorname{Im}\{k\} \Bigg(|k|^2 \int_{G_\rho} |u|^2 dV \tag{5.9}$$

$$+ \int_{G_\rho} |\nabla u|^2 dV + \int_{\substack{\cup F_j \\ F_j \in \mathcal{H}}} \beta_j |u|^2 dS \Bigg) = o(1)$$

Since all the terms of (5.9) are non-negative, it follows that if Im $k > 0$, then $\int_G |u|^2 dV = 0$, whence $u=0$. If Im $k=0$, then we have

$$\lim_{\rho\to\infty} \int_{\Sigma(\rho,0)} |u|^2 dS = 0 \tag{5.10}$$

But by Rellich's well-known "growth estimate",* (5.10) implies $u \equiv 0$ in $\bar{G}$, which proves the theorem

6. Extension to a more general class of finite boundaries.

Certain extensions of Theorem 2.1 with regard to the class of admissible boundaries suggest themselves immediately. For example, the single closed surface B could obviously be replaced by a finite number of regular closed surfaces, exterior to one another, with no essential change in the proof. In this section, we present an extension to a much more general class $\mathcal{H}$ of domains with finite boundaries; $\mathcal{H}$ includes domains bounded by two sided surfaces, e.g. the domain G_0, consisting of all points which are neither on the "horizontal" unit disk $x_1^2 + x_3^2 \leq 1$, $x_2 = 0$, nor on the "vertical" half

*This result, first proved in [1], is a special case of a more general result for "eventually conical" regions which will be proved in Section 7.

disk $x_1^2 + x_2^2 \leq 1$, $x_2 \geq 0$, $x_3 = 0$.

We define $\mathcal{H}$ as follows: $G \epsilon \mathcal{H}$ if $\dot{G}$ is the union of a finite set of $C^{(2+\lambda)}$ surface elements satisfying the conditions

(a) points common to two or more surface elements lie on their edges;

(b) there are no points of G inside a zero angle formed by two surface elements tangent at a common point.*

At a given boundary point $\hat{x}$ of a domain in $\mathcal{H}$, there may be several different boundary conditions, depending on how $\hat{x}$ is approached from the interior of G. In the case of G_o, described above, the origin may have a Neumann condition $\partial u/\partial n = 0$ assigned to it on the "bottom" of the full disk, (i.e. as the origin is approached from the side $x_2 < 0$), and two boundary values assigned to it corresponding to the other directions of approach.

This situation is easily handled if the following definitions, based on a suggestion of Courant's**, are adopted. Here G may be any domain in E^3; it need not belong to $\mathcal{H}$.

We define a restricted type of convergence for sequences x_n of points in G by requiring that x_n and x_m be joined in G by a polygon of arbitrarily small diameter for n and m sufficiently large. A convergent sequence whose limit point is not in G is said to define an <u>oriented boundary point</u>. Two sequences (x_n), (x_n'), each convergent to a point of $\dot{G}$, are said to define the same oriented boundary point if the mixed sequence of points $x_1, x_1', x_2, x_2', \ldots, x_n, x_n', \ldots$ is also convergent. Thus, with every point of $\dot{G}$ we associate one or more oriented boundary points each corresponding to a

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* Note that by this definition $\mathcal{H}$ includes bounded domains.

** See [26], p. 30.

set of convergent sequences which are equivalent in this sense.* The set of oriented boundary points of G will be denoted by ∂G, as contrasted with G, but we shall denote the set $G \cup \partial G$ by the same symbol used for the ordinary closure of G, namely $\bar{G}$; the meaning will generally be clear from the context.

For each $\hat{x} \in \partial G$, corresponding to the convergent sequence (x_n) in G, and $\epsilon > 0$, we denote by $N_\epsilon(\hat{x})$ the following subset of $\bar{G}$: the point y, determined by the convergent sequence (y_n) in G, belongs to $N_\epsilon(\hat{x})$ if x_n and y_n can be joined in G by a polygon of diameter $\leq \epsilon_1 < \epsilon$ for n sufficiently large.

We now define continuity on the boundary ∂G and related concepts (cf. Definitions 1 and 2 of Section 2) in terms of the topology on $\bar{G} = G \cup \partial G$ which has as a base the sets $N_\epsilon(\hat{x})$ together with the ordinary spheres in G. It is evident that these definitions agree with our intuition in regard to continuity at a two-sided surface, etc.

Next, we extend the definition of $\tau_{\bar{G}}(\rho,x)$ to the case where $\bar{G} = G \cup \partial G$. For $x \in G, \tau_{\bar{G}}(\rho,x)$ is defined as the union of $\tau_G(\rho,x)$ with the set of points $\hat{y} \in \partial G$ belonging to $\sigma(x,\rho)$ and such that every neighborhood of $\hat{y}$ (in the new sense) intersects $\tau_G(\rho,x)$. For $\hat{x} \in \partial G$, $\tau_{\bar{G}}(\rho,\hat{x})$ is defined as the union of that component τ_1 of $\sigma(\rho,\hat{x}) \cap G$ which contains a neighborhood of $\hat{x}$, with the set of points $\hat{y} \in \partial G$ belonging to $\sigma(\rho,\hat{x})$, and such that every neighborhood of $\hat{y}$ intersects τ_1.

Finally, we define the analogue in ∂G of a $C^{(2+\lambda)}$ surface element in $\dot{G}$. A <u>$C^{(2+\lambda)}$ face</u> F of ∂G is a subset of ∂G with the following properties.

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* It is simpler, formally, to consider the space of all sets of equivalent sequences convergent in the restricted sense, as in the Cantor method of completing a space. Obviously, there is a one-one correspondence between interior points of G and the sets of equivalent sequences convergent to interior points. The only "new" points which occur are the oriented boundary points.

(a) the set of ordinary points of $\dot{G}$ associated with F constitutes a $C^{(2+\lambda)}$ surface element F_1;

(b) every inner point of F (i.e. points in $\dot{G}$ associated with inner points of F_1) has a neighborhood which does not intersect $\partial G - F$;

(c) all the points of F "lie on the same side of ∂G"; i.e. any two points $\hat{x}_1, \hat{x}_2$ of F may be connected by a continuous curve Γ on F which, with the possible exception of the endpoint $\hat{x}_1$ or $\hat{x}_2$, consists entirely of inner points of F; (a curve on F is a subset of F whose associated points on F_1 form a curve in the ordinary sense, and which is connected in the above topology on $G \cup \partial G$.)

With these definitions the proofs of Sections 3, 4 and 5 hold with no essential change, and, consequently, we obtain

Theorem 6.1 For G in $\mathscr{H}$ and unbounded, let u, defined in $\bar{G} = G \cup \partial G$, satisfy conditions (a) through (d) of Theorem 2.1, where in condition (c) the surface B is replaced by ∂G, and the phrase "$C^{(2+\lambda)}$ surface element" is replaced by "$C^{(2+\lambda)}$ face". Then for $x \in \bar{G}$, $u(x) \equiv 0$.

7. Extension to infinite boundaries and to boundaries with conical points.

If Im $k > 0$, Theorem 6.1 is easily extended to include a general class of domains with infinite boundaries. In fact, we have

Theorem 7.1. Let G be a domain such that for all sufficiently large ρ, $G \cap \sigma(\rho, 0) \in \mathscr{H}$. Let u be a function defined in $\bar{G} = G \cup \partial G$ which satisfies conditions (a) through (d) of Theorem 2.1 with the following modifications: in condition (b), Im $k > 0$; in condition (c), the surface B is to be replaced by $\partial G \cap \bar{\sigma}(\rho, 0)$, and the phrase "$C^{(2+\lambda)}$ surface element" is to be replaced by "$C^{(2+\lambda)}$ face", the resulting condition being assumed for all sufficiently large ρ; finally, condition (d), the radiation condition, is to be changed to

$$\lim_{\rho\to\infty} \int_{\sum(\rho,0)\cap\bar{G}} \left|\frac{\partial u}{\partial r} - iku\right|^2 dS = 0 \tag{7.1}$$

Then for all $x\epsilon\bar{G}$, $u(x) \equiv 0$.

Proof: Throughout Section 5, which includes the proof of Theorem 2.1 (and, essentially, Theorem 6.1) we replace $\sum(\rho,0)$ by $\sum(\rho,0)\cap\bar{G}$. Then as in the case of finite boundaries, (5.9) and the hypothesis Im $k > 0$ imply that $\int_G |u|^2 dV = 0$, whence $u \equiv 0$.*

Remark: The hypotheses of Theorem 7.1 can obviously be weakened; we need know only that $G \cap \sigma(\rho_n, 0) \in \mathcal{H}$ for a sequence (ρ_n) increasing monotonically to infinity, with ρ_n replacing ρ in the other conditions.

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* It should be pointed out that Theorem 4.1, which is essential to the proof of Lemma 5.1, does not apply, as it stands, to the present situation. Theorem 4.1 deals with a complete set of edge points $\mathfrak{E}$ and with a wave function which satisfies the homogeneous conditions of Theorem 2.1 over the entire boundary B. However, some of these conditions were retained merely to simplify the statement of the theorem and the presentation of its proof. An examination of the proof will show that the theorem can easily be extended so as to be applicable here: instead of $\mathfrak{E}$, we consider any subset $\mathfrak{E}' \subset \mathfrak{E}$; then condition (c) of Theorem 2.1 can be weakened by replacing B by B', where B' is that part of ∂G which is contained in some neighborhood of $\mathfrak{E}'$. (G may be any domain, finite or infinite, in $\mathcal{H}$.)

When k is real, the above proof breaks down; in fact, it is known[2] that without further restrictions on the boundary, the solution need not be unique. We shall prove uniqueness for real k for a subset of the class of domains of Theorem 7.1, namely those which are "eventually conical".

A domain G is said to be <u>eventually conical with apex at x_0</u> if for some $\hat{\rho} > 0$, $G - \sigma(\hat{\rho}, x_0)$ is generated by a radially outward directed ray whose endpoint varies over an open set Υ on $\sum(\hat{\rho}, x_0)$. A function u, defined in the closure $\bar{G} = G \cup \partial G$ of such a domain, is said to satisfy <u>eventually conical homogeneous boundary conditions</u> on ∂G if the following is true: there exists a $\rho > \hat{\rho}$ such that at every point $\hat{x}$ in $\partial G - \bar{\sigma}(\rho, x_0)$ either $u = 0$, or else $\lim_{\substack{x \to \hat{x} \\ x \in G}} \nabla u$ exists and $\partial u/\partial n = 0$, where the same boundary condition obtains at every point of a radially outward directed ray included in $\partial G - \bar{\sigma}(\rho, x_0)$.

Closely related to the boundary value problem for the wave equation in an eventually conical region with eventually conical homogeneous boundary conditions is the following <u>associated eigenvalue problem</u> on the unit sphere.* Let Ω be the projection, along radii, of the set Υ on $\sum(\hat{\rho}, x_0)$ onto the unit sphere $\sum(1, x_0)$. (We refer here, and in the next sentence, to the Υ, $\hat{\rho}$ and ρ which occur in the above definitions of eventually conical domains and boundary conditions.) To each boundary point of $\partial\Omega$** we assign the boundary condition $V = 0$ or $\partial V/\partial n = 0$, according as the Dirichlet or Neumann boundary condition obtains on that part of the ray

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* As will be seen below, this eigenvalue problem results from "separating out" the radial part of the wave equation.

** The notion of oriented boundary points can obviously be extended to spherical surface domains and need not be defined formally.

through $\hat{\omega}$ which lies in $G - \sigma(\rho,x_o)$. For these boundary conditions we seek a complete orthonormal set of eigenfunctions V_n with associated eigenvalues λ_n, continuous in $\bar{\Omega}$, and satisfying the equation

$$\Delta^* V_n + \lambda_n V_n = 0 , \qquad \omega \in \Omega . \tag{7.2}$$

Here Δ^* is the Beltrami operator on the sphere - the angular part of the Laplacian - and is given in polar coordinates by

$$\Delta^* = \frac{1}{\sin\Theta}\,\frac{\partial}{\partial\Theta}\left(\sin\Theta\,\frac{\partial}{\partial\Theta}\right) + \frac{1}{\sin^2\Theta}\,\frac{\partial^2}{\partial\phi^2} \tag{7.3}$$

This eigenvalue problem will be called <u>regular</u> if the boundary and boundary conditions have the following properties:

(a) $\partial\Omega$ consists of a finite number of simple $C^{(2+\lambda)}$ arcs,** any two of which have at most endpoints in common, where the interior angle between two arcs at a common endpoint is never zero;

(b) on each arc, one of the boundary conditions $V=0$ or $\partial V/\partial n = 0$ obtains throughout. It is easy to show that if G and u satisfy the condition of Theorem 7.1, where G is an evenually conical domain and u satisfies eventually conical homogeneous boundary conditions on ∂G, then the associated eigenvalue problem is regular.

We first prove an extension of the Rellich growth estimate to wave functions in eventually conical domains.

**A $C^{(m+\lambda)}$ arc is a curve $x_i = \phi_i(s)$, s being the arc length, with $0 \leqq s \leqq \ell$, where $\phi_i \in C^{(m+\lambda)}([0,\ell])$. These arcs may be "one-sided", i.e. they may correspond to $C^{(m+\lambda)}$ faces, enabling us to deal with such boundaries as two-sided slits, etc. If the curve $x_i = \phi_i(s)$ is closed, then it will be called a <u>closed $C^{(m+\lambda)}$ curve</u> provided that $\phi_i^{(m)}(0+) = \phi_i^{(m)}(\ell-)$.

Lemma 7.1. Let G be an eventually conical domain with apex at the origin. Let u be a solution of the wave equation (2.4) with real k which is continuous in $\bar{G}$ and satisfies eventually conical homogeneous boundary conditions on ∂G. Moreover, suppose that the associated eigenvalue problem is regular, and that there exists for this problem a complete orthonormal set of eigenfunctions $V_n(\omega)$. Under these conditions, if

$$\lim_{\rho\to\infty} \int_{\Sigma(\rho,o)\cap G} |u|^2 dS = 0, \tag{7.4}$$

then $u \equiv 0$ in $\bar{G}$.

Proof: Let r_o be $> \hat{\rho}$, ρ, where $\hat{\rho}$ and ρ are the numbers which occur in the above definitions of eventually conical domains and boundary conditions. Then if $x \in G$ and $r = |x| \geq r_o$, we can represent x as $r\omega$, where ω is a point in the set Ω of the associated eigenvalue problem (7.2); (i.e. we are actually using the symbol "ω" to represent the (unit) radius vector to a point of Ω). In terms of the coordinates r, ω, the wave equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\Delta^* u + k^2 u = 0 . \tag{7.5}$$

Let us enclose each corner ω_i of $\partial\Omega$ by a small circle $\Gamma(\epsilon,\omega_i)$ of radius ϵ and center ω_i and let Ω_ϵ be the part of Ω outside these circles. The term "corner" will also be used for any point at which the boundary conditions are discontinuous. We now multiply (7.5) by any eigenfunction $V_n(\omega)$ and integrate over Ω_ϵ, obtaining, for $r \geq r_o$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\int_{\Omega_\epsilon} u\, V_n d\omega\right) + \int_{\Omega_\epsilon} V_n \Delta^* u d\omega + \kappa^2 r^2 \int_{\Omega_\epsilon} u V_n d\omega = 0 \tag{7.6}$$

The change in the order of differentiation and integration in the first term is permissible, since $u(r\omega)$ is a $C^{(2+\lambda)}$ function of x,y,z, and hence r, in any region included in $[r_o,\infty]\times\overline{\Omega}_\epsilon$. (This follows from Theorems I-1, 3.2 and 3.4; these theorems are applicable, since if Γ is a $C^{(2+\lambda)}$ arc on $\partial\Omega$ and $r_o < r'$, then the set of points (r,ω) belonging to $[r_o,r'] \times \Gamma$ is a $C^{(2+\lambda)}$ face on ∂G.)

Now we apply Green's theorem** for the self adjoint operator $-\Delta^*$ to the second integral in (7.6), and then make use of (7.2), obtaining

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\int_{\Omega_\epsilon} uV_n d\omega\right)-\lambda_n\int_{\Omega_\epsilon} uV_n d\omega + k^2r^2\int_{\Omega_\epsilon} uV_n d\omega = \int_{\partial\Omega_\epsilon}\left(V_n\frac{\partial u}{\partial n} - u\frac{\partial V_n}{\partial n}\right) ds \tag{7.7}$$

** Both forms of Green's theorem are obtained in the same way as the corresponding theorems in euclidean n-space - we use integration by parts, applied to the integral over Ω_ϵ of $V_n\Delta^* u$ in polar coordinates

$$\iint_{\Omega_\epsilon} V_n\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\right]\sin\theta d\theta d\phi$$

Note that by the hypothesis, Ω, and hence Ω_ϵ, consist of at most a finite number of separate domains for which Green's theorem is easily shown to be valid. It is necessary, of course, that the functions involved be sufficiently smooth, say, possessed of continuous second derivatives on Ω_ϵ; this has just been shown for u, and it follows for V_n by the application of Theorems 3.2 and 3.4 to the wave function w defined in (7.9) below.

Next, we integrate (7.7) with respect to r, from r_o to r, ($r_o<r$), divide by r^2, and integrate again from r_o to r. We get

$$\int_{\Omega_\epsilon} u(r\omega)V_n d\omega - \int_{\Omega_\epsilon} u(r_o\omega) - \int_{r_o}^{r} d\xi \frac{r_o^2}{\xi^2} \int_{\Omega_\epsilon} u_r(r_o\omega)V_n d\omega$$

$$+ \int_{r_o}^{r} \frac{d\eta}{\eta^2} \int_{r_o}^{\eta} d\xi \left[(k^2\xi^2-\lambda_n) \int_{\Omega_\epsilon} u(\xi\omega)V_n d\omega\right]$$

$$= \int_{r_o}^{r} \frac{d\eta}{\eta^2} \int_{r_o}^{\eta} d\xi \int_{\partial\Omega_\epsilon} (V_n \frac{\partial u(\xi\omega)}{\partial n} - u(\xi\omega)\frac{\partial V_n}{\partial n})ds \qquad (7.8)$$

Now let $\epsilon \to 0$. The line integral over $\partial\Omega_\epsilon$ tends to zero uniformly with respect to r. For by Theorem 4.1 (as modified in the footnote to the proof of Theorem 7.1) we see that as $\epsilon \to 0$, $\epsilon \left.\frac{\partial u}{\partial n}\right|_{\Gamma(\epsilon,\omega_i)} \longrightarrow 0$ uniformly with respect to r and ω in $[r_o,r'] \times \Gamma(\epsilon,\omega_i)$, $(r' > r_o)$. We prove that $\epsilon \left.\frac{\partial V_n}{\partial n}\right|_{\Gamma(\epsilon,\omega_i)} \longrightarrow 0$ uniformly in r and ω by applying the same theorem to the function

$$w(r\omega) = j_{\nu_n}(kr)V_n(\omega) = \sqrt{\frac{\pi}{2kr}}\; J_{\nu_n+1/2}(kr)V_n(\omega) \qquad (7.9)$$

Here $\lambda_n = \nu_n(\nu_{n+1})$; thus w is a solution of (7.5) which satisfies the boundary conditions on $\partial G\text{-}\sigma(r_o,0)$. Again by Theorem 4.1, the integral $\int_{\Omega_\epsilon} u_r(r_o\omega)V_n d\omega$ in the third term of (7.8) approaches a finite constant c_o as $\epsilon \to 0$. Finally, the integral $\int_{\Omega_\epsilon} u(r\omega)V_n d\omega \to \int_{\Omega} u(r\omega)V_n d\omega$ uniformly in any interval $[r_o,r']$; this follows from the uniform continuity of u in $[r_o,r'] \times \Omega$. In the resulting equation, let us set

$$v_n(r) = \int_\Omega u(r\omega)V_n(\omega)d\omega \qquad\qquad r \geqq r_o \tag{7.10}$$

We get

$$v_n(r) + \int_{r_o}^{r} \frac{d\eta}{\eta^2} \int_{r_o}^{\eta} d\xi\left[(k^2\xi^2-\lambda_n)v_n(r)\right] = v_n(r_o) - c_o r_o^2\left(\frac{1}{r} - \frac{1}{r_o}\right) \tag{7.11}$$

Upon differentiating (7.11), multiplying the result by r^2, and differentiating again, we get

$$\frac{d}{dr}\left[r^2 \frac{d}{dr} v_n(r)\right] + \left[k^2r^2 - \lambda_n\right]v_n(r) = 0 \tag{7.12}$$

Setting $\lambda_n = \nu_n(\nu_n+1)$, we see that $\nu_n(r)$, is a linear combination of the spherical Bessel and Neumann functions j_{ν_n} and n_{ν_n}, respectively of order ν_n and argument kr.

Now, suppose $u(r\omega) \neq 0$ for some $\omega\epsilon\Omega$ and $r \geqq r_o$. Then, by the continuity of u and completeness of the set of eigenfunctions $V_n(\omega)$, there exists an m such that

$$v_m(r) = \int_\Omega u\, V_m d\omega = c_1 j_{\nu_m}(kr) + c_2 n_{\nu_m}(kr) \neq 0 \tag{7.13}$$

Here c_1 and c_2 are complex numbers which are not both zero. We have, as $r \to \infty$

$$\nu_m(r) \sim \frac{c_1}{kr} \sin\left[kr - (\nu_m+1)\pi/2\right] + \frac{c_2}{kr} \cos\left[kr - (\nu_{m+1})\pi/2\right]$$

$$\sim \frac{b_1}{r} \sin(kr+\beta_1) + i\, \frac{b_2}{r}\sin(kr+\beta_2) \tag{7.14}$$

where b_i and β_i are real, $i = 1,2$, and b_1 and b_2 are not both zero.

By a simple application of Bessel's inequality,

$$\int\limits_{\Sigma(r,0)\cap G} |u|^2 dS = r^2 \int\limits_{\Omega} |u(r\omega)|^2 d\omega \geqq r^2 |v_m(r)|^2 \sim b_1^2 \sin^2(kr + \beta_1) + b_2^2 \sin^2(kr + \beta_2) \tag{7.15}$$

But this contradicts (7.4). Hence our assumption that $u(r\omega) \neq 0$ for some $\omega \epsilon \Omega$ and $r \geqq r_o$ is untenable, and $u \equiv 0$ for $r\omega$ in the conical part $[r_o, \infty] \times \Omega$ of G. Since u is analytic in the domain G, this implies that $u \equiv 0$ throughout $\bar{G}$.*

The application of this extended growth estimate to the proof of uniqueness in eventually conical domains when k is real is seen to depend essentially on the existence of a complete set of eigenfunctions for the associated eigenvalue problem. In Appendix III we prove** the existence of a complete set of eigenfunctions whenever the eigenvalue problem is regular and the boundary conditions have the following additional property, which we shall refer to as property P:

P. If Ξ is a multiply connected component of Ω, the same boundary condition, (e.g. $u = 0$), holds on all of $\partial\Xi$, with the possible exception of one connected component of $\partial\Xi$; on some parts, but not all, of the latter component, the other boundary condition, (e.g. $\partial u/\partial n = 0$) may hold. Thus we can have mixed boundary conditions on at most one component of the boundary of any multiply connected component of Ω.

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* The basic idea of this proof is similar to that of Rellich's [1] for finite boundaries, where the eigenfunctions for the complete sphere are used.

** Actually this result (part of Theorem III-1) is established for the case where Ω is connected, i.e. Ω is a domain on $\Sigma(1,0)$; however, it can obviously be extended with no difficulty to any regular eigenvalue problem, since Ω then has at most a finite number of connected components.

Theorem 7.2 Let G and u satisfy the conditions of Theorem 7.1 except that Im k = 0. In addition suppose that G is eventually conical, that u satisfies eventually conical homogeneous boundary conditions on ∂G, and that the associated (regular) eigenvalue problem possesses property P.

Then $u \equiv 0$ in $\bar{G}$.*

Proof: The proof of Theorem 7.1, when applied to the case of real k, yields (7.4). Then, since Theorem III-1 guarantees the existence of a complete set of eigenfunctions under the hypothesis, Lemma 7.1 yields the result.

The uniqueness theorems proved up to now do not apply to domains with such simple boundaries as the surface of a finite cone; this surface cannot be subdivided into $C^{(2+\lambda)}$ surface elements. However such singularities of the boundary as the apex of a cone can be handled by a method similar to that used for eventually conical domains.

If G is a domain, a point $x_0 \in \partial G$ is said to be a conical boundary point if, for some $\hat{\rho} > 0, G \cap \bar{\sigma}(\hat{\rho}, x_0)$ is generated by a line segment with one endpoint fixed at x_0 and the other endpoint varying over an open set T on $\sum(\hat{\rho}, x_0)$. A function u defined in $\bar{G}$ is said to satisfy conical homogeneous boundary conditions near the conical boundary point x_0 if the following is true: there exists a $\rho < \hat{\rho}$ such that at every point x of $\partial G \cap \bar{\sigma}(\rho, x_0)$ either $u = 0$, or else $\lim\limits_{\substack{x \to \hat{x} \\ x \in G}} \nabla u$ exists and $\partial u / \partial n = 0$, where the same boundary condition obtains at every point of a radial line segment included in $\partial G \cap \bar{\sigma}(\rho, x_0)$. Exactly as in the case of eventually conical domains, we consider the projection Ω of T, along radii, on the unit sphere $\sum(1, x_0)$, and the associated eigenvalue problem (7.2).

*Odeh, ([29], p. 56) has recently proved uniqueness (for real k) for a class of boundaries which are not necessarily eventually conical. The weaker assumption is made that for r sufficiently large, $\partial r / \partial n \leq 0$ on the boundary. However, this result is restricted to smooth boundaries on which the boundary condition $u = 0$ holds.

<u>Theorem 7.3</u> Let x_o be a conical boundary point of a domain G. Let u be a solution of the wave equation (2.4) which is continuous in $\bar{G}$, and satisfies conical homogeneous boundary conditions near x_o. Moreover, suppose that the associated eigenvalue problem is regular and has property P. Then in a neighborhood of x_o, u can be expanded in a uniformly and absolutely convergent series:

$$u = \sum_{n=1}^{\infty} a_n j_{\nu_n}(kr) V_n(\omega) \tag{7.16}$$

Here $r = |x-x_o|$ and $\nu_n(\nu_{n+1}) = \lambda_n$, with $\nu_n \geq 0$, λ_n being the eigenvalue corresponding to the eigenfunction $V_n(\omega)$ of the associated eigenvalue problem. Moreover, the series (7.16) can be differentiated term by term with respect to r, the resulting series also being uniformly and absolutely convergent in the neighborhood of x_o.

<u>Remark:</u> The first few terms of the differentiated series may, of course, become infinite as $r \to 0$.

<u>Proof:</u>* Let r_o be $< \hat{\rho}, \rho$ where $\hat{\rho}$ and ρ are the numbers appearing in the paragraph preceding this theorem. As in the proof of Lemma 7.1, we represent x in $G \cap \sigma(r_o, x_o)$ by $r\omega$, where ω is in the set Ω of the associated eigenvalue problem.

The proof of Lemma 7.1 yields the fact that $v_n(r)$, defined by (7.10), is a linear combination of the spherical Bessel and Neumann functions of kr of order ν_n. But since u is continuous for $r = 0$, we have

$$v_n(r) = a_n j_{\nu_n}(kr) \quad , \qquad r \leq r_o \, . \tag{7.17}$$

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* This proof is based, essentially, on that given by Müller ([27], Theorem 13) for the expansion of a wave function near an interior point. Similarly, the proof of Theorem 8.1, dealing with the expansion for large r of a wave function in an eventually conical domain, is based on the proof of Theorem 16 of [27], which gives the expansion in the simple case where ∂G is finite.

From the completeness of the eigenfunctions $V_n(\omega)$, we have

$$\int_{\Omega} |u(r\omega)|^2 d\omega = \sum_{n=1}^{\infty} |v_n(r)|^2 < \infty \quad , \quad r \leqq r_o \; . \tag{7.18}$$

From this it follows that for some constant A

$$|a_n j_{\nu_n}(kr_o)| \leqq A \; , \qquad n = 1,2,\ldots \tag{7.19}$$

In Theorem III-1, in addition to establishing the existence of the complete orthornormal set of eigenfunctions $V_n(\omega)$ for the mixed boundary eigenvalue problem being considered, we prove the following facts. First, the eigenvalues are all non-negative, $(0 \leqq \lambda_1 \leqq \lambda_2 \leqq \ldots)$ and, if p is the least integer such that $\lambda_p > 0$, then

$$\sum_{n=p}^{\infty} \lambda_n^{-2} < \infty \tag{7.20}$$

Second, there exists a positive constant Q such that

$$|V_n(\omega)| \leqq Q\lambda_n^2 \; , \qquad n \geqq p \; , \qquad \omega \epsilon \Omega \tag{7.21}$$

Now, from (7.19) and (7.21) we get

$$|a_n j_{\nu_n}(kr) V_n(\omega)| \leqq AQ\lambda_n^2 \left| \frac{j_{\nu_n}(kr)}{j_{\nu_n}(kr_o)} \right| \; . \tag{7.22}$$

Next, from the power series expansion of $J_{\nu_n + 1/2}(kr)$, we get as $n \to \infty$

$$\frac{j_{\nu_n}(kr)}{j_{\nu_n}(kr_o)} \sim \left(\frac{r}{r_o}\right)^{\nu_n} \tag{7.23}$$

uniformly in the interval $0 \leq r \leq r_o$. Therefore, for n sufficiently large, we have

$$\left|\frac{j_{\nu_n}(kr)}{j_{\nu_n}(kr_o)}\right| < 2\left(\frac{r}{r_o}\right)^{\nu_n} \qquad r \leq r_o . \tag{7.24}$$

From (7.20) it follows that

$$\sum_{n=p}^{\infty} \nu_n^{-4} < \infty \tag{7.25}$$

Since ν_n is monotone increasing with n, (7.25) implies that for all n sufficiently large $\nu_n > n^{1/5}$. Hence there exists a constant $C_1 > 0$ such that

$$\nu_n \geq C_1 n^{1/5} , \qquad n \geq p \tag{7.26}$$

Now let q be any integer ≥ 0, and let $0 \leq \alpha \leq \alpha_o < 1$. For some $C_2 = C_2(\alpha_o)$ $\nu_n^q \alpha^{\nu_n/2} \leq C_2$ for all n, and we have

$$\sum_{n=1}^{N} \nu_n^q \alpha^{\nu_n} \leq C_2 \sum_{n=1}^{N} \alpha^{\nu_n/2} \leq C_2 \sum_{k=o}^{[\nu_N]} \sum_{k \leq \nu_n < k+1} \sqrt{\alpha}^{\nu_n}$$

$$\leq C_2 \sum_{k=o}^{[\nu_N]} C_1^{-5}(k+1)^5 \sqrt{\alpha}^k . \tag{7.27}$$

Thus

$$\sum_{n=1}^{\infty} \nu_n^q \alpha^{\nu_n} \leq C_3 < \infty \qquad\qquad 0 \leq \alpha \leq \alpha_o . \tag{7.28}$$

From (7.28) we see that the series $\sum_{n=1}^{\infty} \lambda_n^2 (\frac{r}{r_o})^{\nu_n}$ is absolutely and uniformly convergent for $0 \leq r \leq r_1$ where $r_1 < r_o$. Therefore, by (7.22) and (7.24), the series $\sum_{n=1}^{\infty} a_n j_{\nu_n}(kr)V_n(\omega)$ is absolutely and uniformly convergent for $0 \leq r \leq r_1$ and $\omega\epsilon\Omega$; hence it represents a continuous function in the indicated range.

In the same way, using the recursion formulas

$$2J'_\nu(z) = J_{\nu-1}(z) - J_{\nu+1}(z) , \tag{7.29}$$

$$\frac{2\nu}{z} J_\nu(z) = J_{\nu-1}(z) + J_{\nu+1}(z) , \tag{7.30}$$

we can show that the series obtained by differentiating (7.16) term by term is absolutely and uniformly convergent for $0 \leq r \leq r_1$ and $\omega\epsilon\Omega$. Thus, if (7.16) is valid, $\partial u/\partial r$ may be obtained by termwise differentiation.

Finally, to show (7.16) is valid, let

$$\Phi(r\omega) = \sum_{n=1}^{\infty} a_n j_{\nu_n}(kr)V_n(\omega) . \tag{7.31}$$

For all V_n, $0 \leq r \leq r_1$, we have, using the uniformity of convergence of (7.31),

$$\int_\Omega [u(r\omega) - \Phi(r\omega)]V_n(\omega)d\omega = 0 \tag{7.32}$$

From the completeness of the set of eigenfunctions V_n, and the continuity of u and Φ , it follows that

$$u(r\omega) \equiv \Phi(r\omega) \qquad 0 \leq r \leq r_1 , \quad \omega\epsilon\Omega , \tag{7.33}$$

which completes the proof.

Theorem 7.3 enables us to prove uniqueness in domains possessing one or more conical boundary points. For simplicity, we first consider the case of a domain with a finite boundary possessing a single conical boundary point.

<u>Theorem 7.4.</u> Let G be a domain with a conical boundary point x_o and the property that for all sufficiently small ϵ, $G - \bar{\sigma}(\epsilon,x_o)$ is the exterior of a regular closed surface. Let u be a function defined in $\bar{G}$ satisfying conical homogeneous boundary conditions near x_o, as well as conditions (a) through (d) of Theorem 2.1, with the following modification: in condition (c), the surface B is to be replaced by $\dot{G} - \sigma(\epsilon,x_o)$, the resulting condition being assumed for all sufficiently small ϵ. In addition, let the associated eigenvalue problem at x_o possess property P.

Then $u \equiv 0$ in $\bar{G}$.

<u>Proof:</u> We shall show that Lemma 5.1 holds in this situation, after which the proof is exactly the same as that of Theorem 2.1.

Let $G^{\epsilon}_{\rho} = G_{\rho} - \bar{\sigma}(\epsilon,x_o)$, where G_{ρ} is as defined in Section 5, namely, $G_{\rho} = G \cap \sigma(\rho,0)$. For ϵ sufficiently small and ρ sufficiently large, the proof of Lemma 5.1 (see also the footnote to the proof of Theorem 7.1) yields the equation

$$\int_{G_\rho^\epsilon} \bar{u}\Delta u dV + \int_{G_\rho^\epsilon} \nabla\bar{u}\cdot\nabla u dV = \int_{\dot{G}_\rho^\epsilon} \bar{u}\,\frac{\partial u}{\partial n}\,dS = \int_{\dot{G}_\rho - \sigma(\epsilon, x_o)} \bar{u}\,\frac{\partial u}{\partial n}\,dS - \int_{\Sigma(\epsilon, x_o)\ G} \bar{u}\,\frac{\partial u}{\partial r}\,dS \tag{7.34}$$

Here $r = |x-x_o|$. We shall prove, using Theorem 7.3, that the last integral approaches zero as $\epsilon \to 0$. Once this is done, the same reasoning as that used in the proof of Lemma 5.1, shows that the remaining integrals approach finite limits given by the corresponding integrals over G_ρ and $\dot{G}_\rho$; this will establish Lemma 5.1 in the present case.

Since the hypotheses of this theorem imply those of Theorem 7.3, we have the expansion (7.16) for $u(r\omega)$ in a neighborhood of x_o. Moreover we can obtain $\frac{\partial u}{\partial r}$ by differentiating (7.16) term by term. Now, except possibly for a finite number of terms at the beginning of the differentiated series, say the first t terms, every term approaches zero, and the series is uniformly convergent. Hence we have in the neighborhood of x_o, as $r \to 0$

$$\frac{\partial u}{\partial r} = k\sum_{n=1}^{t} a_n j'_{\nu_n}(kr)V_n(\omega) + o(1) \tag{7.35}$$

uniformly in ω. Now let $\tilde{\nu}$ be the least order > 0 of the spherical Bessel functions appearing in (7.35). Then it is easy to see, using (7.29) and (7.30), that

$$\frac{\partial u}{\partial r} = O(r^{\tilde{\nu}-1}) + o(1) \tag{7.36}$$

uniformly in ω.

From (7.36) we see immediately that

$$\lim_{\epsilon\to 0} \int_{\sum(\epsilon,x_0)\cap G} \bar{u} \frac{\partial u}{\partial r} dS = 0 \tag{7.37}$$

which proves the theorem.

It is evident that each of the uniqueness theorems proved in this paper - Theorems 2.1, 6.1, 7.1 and 7.2 - can easily be extended in a similar way so as to admit a finite number of conical boundary points. (In the case of Theorem 7.1, a denumerable set of such points may be admitted.)

8. Supplementary results.

In addition to the uniqueness theorems, the methods of Section 7 yield a number of results which are useful in connection with certain problems in diffraction theory. These results are principally concerned with the behavior of a wave function for large r, in an eventually conical domain.

Theorem 8.1. Let G be an eventually conical domain with apex at the origin. Let u be a solution of the wave equation (2.4) which is continuous in $\bar{G}$, satisfies eventually conical homogeneous boundary conditions on ∂G and satisfies the radiation condition (7.1). In addition, suppose that the associated eigenvalue problem is regular and possesses property P.

Then there exists an r_1 such that

$$u = \sum_{n=1}^{\infty} b_n h_{\nu_n}(kr) V_n(\omega) , \qquad r \geq r_1 , \quad \omega \in \Omega \tag{8.1}$$

where the convergence is absolute and uniform over any compact subset of the indicated range.

Here $r = |x|$, and ν_n, V_n, and Ω have the same meaning as in Theorem 7.3; h_{ν_n} is the spherical Hankel function given by

$$h_{\nu_n}(kr) = \sqrt{\frac{\pi}{2kr}}\, H^{(1)}_{\nu_{n+1/2}}(kr) \tag{8.2}$$

Proof: Let r_o be defined as in the proof of Lemma 7.1. It follows from the latter proof that, for $r \geq r_o$, $v_n(r)$, defined by (7.10), is a linear combination of the spherical Hankel and Bessel functions:

$$v_n(r) = b_n h_{\nu_n}(kr) + c_n j_{\nu_n}(kr) \ . \tag{8.3}$$

We also have

$$\frac{dv_n(r)}{dr} = \int_\Omega \frac{\partial u}{\partial r} V_n(\omega)d\omega \tag{8.4}$$

The change in the order of differentiation and integration is justified by the method used in the proof of (7.12).

From the radiation condition (7.1) we have

$$\int_\Omega \left|\frac{\partial u}{\partial r} - iku\right|^2 d\omega = o(r^{-2}) \ . \tag{8.5}$$

Using (8.4), the Cauchy-Schwartz inequality, and (8.5) we get

$$\left|\frac{dv_n(r)}{dr} - ikv_n(r)\right| = \left|\int_\Omega (\frac{\partial u}{\partial r} - iku)\ V_n(\omega)d\omega\right|$$

$$\leq \left[\int_\Omega \left|\frac{\partial u}{\partial r} - iku\right|^2 d\omega \cdot \int_\Omega |V_n(\omega)|^2 d\omega\right]^{1/2} = o(r^{-1}) \tag{8.6}$$

But it is well known that, for Im $k \geq 0$,

$$\frac{d}{dr} h_\nu(kr) - ikh_\nu(kr) = O(r^{-2}) \ , \tag{8.7}$$

while

$$\frac{d}{dr} j_\nu(kr) - ikj_\nu(kr) = \frac{e^{-i(kr-\nu\pi/2)}}{r} + O(r^{-2}) \tag{8.8}$$

It follows from (8.3), (8.6), (8.7) and (8.8) that $c_n = 0$, and we have, therefore

$$v_n(r) = b_n h_{\nu_n}(kr). \tag{8.9}$$

Next, from the power series expansion of $H_{\nu_n+1/2}(kr)$,* we get for fixed r, as $n \to \infty$,

$$h_{\nu_n}(kr) \sim \frac{\Gamma(\nu_n+1/2)}{2\sqrt{\pi}\, i} \left(\frac{2}{kr}\right)^{\nu_n+1} \tag{8.10}$$

uniformly with respect to r in any finite interval $0 < \alpha \leqq r \leqq \beta < \infty$. From (8.10) we get an estimate analogous to (7.23); namely, as $n \to \infty$ we have for fixed r

$$\frac{h_{\nu_n}(kr)}{h_{\nu_n}(kr_o)} \sim \left(\frac{r_o}{r}\right)^{\nu_n+1} \tag{8.11}$$

uniformly with respect to r in the finite interval $r_o \leqq r \leqq r' < \infty$.

As in the proof of Theorem 7.3 we can show that there exists a constant A such that

$$|b_n h_{\nu_n}(kr_o)| < A . \qquad n = 1,2,\ldots \tag{8.12}$$

From this point on the proof is essentially the same as that of Theorem 7.3, and will therefore be omitted.

<u>Theorem 8.2.</u> Let G and u satisfy the hypothesis of Theorem 8.1. Then as $r \to \infty$,

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* This is obtained from the expansions for $J_{\nu+1/2}(z)$ and $N_{\nu+1/2}(z)$ given, for example, in [28], pp. 16, 17.

$$u(r\omega) = \frac{e^{ikr}}{r} f(\omega) + o\left(\frac{e^{ikr}}{r}\right) , \tag{8.13}$$

$$\frac{\partial u(r\omega)}{\partial r} = ik \frac{e^{ikr}}{r} f(\omega) + o\left(\frac{e^{ikr}}{r}\right) \tag{8.14}$$

The estimates of the remainder terms, $o(e^{ikr}/r)$, are uniform with respect to all "directions" ω in $\bar{\Omega}$. $f(\omega)$ is a continuous function on $\bar{\Omega}$.

Proof: In each term of the series in (8.1) we represent $h_{\nu_n}(kr)$ by means of Sommerfeld's integral:

$$h_{\nu_r}(kr) = \frac{1}{\sqrt{2\pi kr}} \int_{L_1} e^{ikr\cos\phi} e^{i(\nu_n+1/2)(\phi-\pi/2)} d\phi , \tag{8.15}$$

where L_1 is a contour in the ϕ-plane from $-\pi/4 + i\infty$ to $\pi/4 - i\infty$.* We get

$$u(r\omega) = \frac{1}{\sqrt{2\pi kr}} \sum_{n=1}^{\infty} \int_{L_1} e^{ikr\cos\phi} b_n e^{i(\nu_n+1/2)(\phi-\pi/2)} V_n(\omega) d\phi . \tag{8.16}$$

We now wish to change the order of summation and integration. By (8.10) and (8.12), there exists a positive constant A_1 such that

$$|b_n| \leq \frac{A_1}{\Gamma(\nu_n+1/2)} \left(\frac{kr_o}{2}\right)^{\nu_n} \qquad n = 1,2,\ldots \quad . \tag{8.17}$$

Next, inequalities (7.21), (7.26) and (8.17) enable us to show, by a method similar to that used in proving (7.28), that for all $\omega\epsilon\Omega$ and all ϕ;

$$\sum_{n=1}^{\infty} \left| b_n e^{i(\nu_n+1/2)(\phi-\pi/2)} V_n(\omega) \right| \leq Me^{\mu} \exp|\mathrm{Im}\ \phi| \tag{8.18}$$

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* With this contour the integral in (8.15) will be convergent for $r > 0$ and for all non-zero k such that $\mathrm{Re}\ k$, $\mathrm{Im}\ k \geqq 0$.

Here M and μ are positive constants depending on Ω, A_1, k and r_o. The convergence is uniform for ω in Ω and ϕ in any bounded region. Thus

$$W(\phi,\omega) = \sum_{n=1}^{\infty} b_n e^{i(\nu_n+1/2)(\phi-\pi/2)} V_n(\omega) \tag{8.19}$$

is analytic in ϕ over the entire complex ϕ-plane for each $\omega\epsilon\Omega$; also, W is continuous in ϕ and ω together for all ϕ and $\omega\epsilon\Omega$. It follows from the uniform convergence of the series in (8.19) that the change in order of summation and integration would be permissible in (8.16) if L_1 were replaced by any finite part of itself. Moreover, we have for all sufficiently large r

$$\int_{L_1} \left| e^{ikr \cos \phi} \; e^{\mu \exp|\mathrm{Im}\ \phi|} \right| d\phi < \infty. \tag{8.20}$$

Hence the integral over L_1 of the series of absolute values of the integrands in (8.16) converges. From these facts it follows that the change in order of summation and integration in (8.16) is justified, and we have, for r sufficiently large,

$$u(r\omega) = \frac{1}{\sqrt{2\pi kr}} \int_{L_1} e^{ikr \cos \phi}\, W(\phi,\omega)d\phi \tag{8.21}$$

A simple application of the method of steepest descents to this integral yields (8.13), where

$$f(\omega) = \frac{1}{k} W(0,\omega) \quad e^{-\pi i/4} \tag{8.22}$$

The uniformity of the error estimate with respect to ω follows from the uniform continuity of W in $\Gamma \times \Omega$, where Γ is a neighborhood of the origin in the ϕ-plane.

To obtain (8.14), we differentiate (8.21), obtaining

$$\frac{\partial u}{\partial r} = \frac{1}{\sqrt{2\pi kr}} \int_{L_1} e^{ikr \cos \phi} ik \cos \phi \, W(\phi,\omega) d\phi - \frac{u}{2r} \tag{8.23}$$

The change in order of differentiation and integration is justified by the fact that the derivative of the integrand with respect to r is everywhere continuous, and by the fact that the integral of the derivative, as well as the original integral in (8.21), is uniformly convergent with respect to r.

We now apply the method of steepest descents to the integral in (8.23), noting that it differs from the integral in (8.21) only in that $W(\phi,\omega)$ is replaced by $ik \cos\phi \, W(\phi,\omega)$; thus the function $f(\omega)$ must be replaced by $ikf(\omega)$ in the result. The uniformity, with respect to ω, of the error term for this integral is established in the same way as that in (8.13). Finally, the remaining term of (8.23), namely $\frac{u}{2r}$, is obviously $o(e^{ikr}/r)$ uniformly in ω, which completes the proof.

It follows from (8.13) and (8.14) that

$$r\left(\frac{\partial u}{\partial r} - iku\right) = o(1) \tag{8.24}$$

uniformly in all directions ω. Thus, in eventually conical regions for which the hypotheses of Theorem 8.1 are valid, the integral form of the radiation condition is equivalent to the form (8.24). In particular, this holds whenever ∂G is finite; this enables us to prove the invariance of the integral form of the radiation condition under a translation of

the center of the spheres on which the integration is performed, as mentioned in the remark at the end of Section 2.

Theorem 8.3. Let G be an infinite domain with a finite boundary, and let u be a solution of the wave equation (2.4) in G, which satisfies the radiation condition (2.5). Then, for any point x_o, we have

$$\lim_{\rho\to\infty} \int_{\sum(\rho,x_o)} \left| \frac{\partial u}{\partial r} - iku \right|^2 dS = 0 \tag{8.25}$$

Proof: By the preceding remarks, it is sufficient to prove the invariance of the radiation condition in the form (8.24), under a translation of the origin to x_o. Let $\hat{r}$ be the distance of a point to x_o, and let $\delta = |x_o|$. We have at any point x

$$\left|\hat{r}\left(\frac{\partial u}{\partial \hat{r}} - iku\right) - r\left(\frac{\partial u}{\partial r} - iku\right)\right| \leq ik\delta|u| + \delta\left|\frac{\partial u}{\partial \hat{r}}\right| + r\left|\frac{\partial u}{\partial r} - \frac{\partial u}{\partial \hat{r}}\right| \tag{8.26}$$

Next, by (8.13), as $\hat{r} \to \infty$, and therefore $r \to \infty$, $u \to 0$ uniformly with respect to direction from x_o. From this, using (3.45), we see also that $|\nabla u| \to 0$, uniformly with respect to direction, as $\hat{r} \to \infty$. Hence the first two terms on the right hand side of (8.26) approach zero uniformly in direction, as $\hat{r} \to \infty$. To handle the remaining term, let ψ be the angle subtended at x by the radius vectors from x_o and from the origin, and let θ be the angle between the latter radius vector and the vector ∇u. We have

$$\left|\frac{\partial u}{\partial r} - \frac{\partial u}{\partial \hat{r}}\right| = |\nabla u|\,|\cos\theta - \cos(\theta+\psi)| = |\nabla u|\,|(1-\cos\psi)\cos\theta + \sin\psi\sin\theta|$$

$$\leq 2|\nabla u|\,|\sin(\psi/2)| \leq 2\,\frac{\delta}{\hat{r}}\,|\nabla u| \tag{8.27}$$

Hence $\hat{r}\left|\frac{\partial u}{\partial r} - \frac{\partial u}{\partial \hat{r}}\right| \longrightarrow 0$ as $\hat{r} \to \infty$.

It follows, therefore, from (8.24) and (8.26) that

$$\lim_{r\to\infty} \hat{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0 \tag{8.28}$$

uniformly with respect to angle. This completes the proof.

We conclude with an application of Theorem 7.3. Let $\hat{x}$ be a point on an edge common to two surface elements on the boundary of a domain G which are plane in a neighborhood of $\hat{x}$, and which form an interior angle β at $\hat{x}$. If $\beta = 2\pi$, we assume that the edge is a straight line in the neighborhood of $\hat{x}$. (For $0 < \beta < 2\pi$, this is, of course, necessarily the case.) Suppose further that either the homogeneous Dirichlet or Neumann boundary condition obtains on each surface element.

We see that $\hat{x}$ is a conical point with conical boundary conditions, and, moreover, that the remaining hypotheses of Theorem 7.3 are satisfied. Hence a wave function u in G can be expanded in a neighborhood of $\hat{x}$ as in (7.16). Let us assume, for definiteness, that the boundary condition $u = 0$ obtains on both surface elements. We shall prove that the orders ν_n of the Bessel functions in this case are of the form $m\pi/\beta + \ell$ where ℓ and m are integers with $\ell \geq 0$, $m > 0$.

Consider any eigenfunction $V_n(\theta,\phi)$ of the associated eigenvalue problem; here θ and ϕ are polar coordinates in terms of which the region Ω of the eigenvalue problem is the lune described by the inequalities $0 < \theta < \pi$, $0 < \phi < \beta$. Multiply (7.2) by $\sin m\pi\phi/\beta$, and integrate with respect to ϕ from 0 to β. Let

$$\hat{T}_{n,m}(\theta) = \int_0^\beta V_n(\theta,\phi) \sin \frac{m\pi\phi}{\beta}\, d\phi . \tag{8.29}$$

Upon setting $z = \cos\Theta$, we find that $T_{n,m}(z) = \hat{T}_{n,m}(\cos^{-1}z)$ satisfies the equation

$$\frac{d}{dz}\left[(1-z^2)\frac{dT}{dz}\right] + \left(\lambda_n - \frac{\mu^2}{1-z^2}\right) T = 0 \ , \quad -1 < z < 1 \tag{8.30}$$

Here $\mu = m\pi/\beta$. Expanding the solution in powers of $z \mp 1$ at the singular points $z = \pm 1$, and noting that $T_{n,m}(z)$ is finite at these points, we find that in the neighborhood of $z = 1$ and $z = -1$ respectively, we have

$$T_{n,m} = (1-z)^{\mu/2}\, v_+(z) \ , \tag{8.31}$$

and

$$T_{n,m} = (1+z)^{\mu/2}\, v_-(z) \ . \tag{8.32}$$

Here $v_+(z)$ and $v_-(z)$ are regular analytic functions in the indicated neighborhoods. It follows that $v(z) = (1-z^2)^{-\mu/2}\, T_{n,m}(z)$ is regular at the points $z = \pm 1$, and hence regular over the entire z-plane. The function $v(z)$ satisfies the differential equation

$$(1-z^2)\frac{d^2v}{dz^2} - 2(\mu+1)\, z\, \frac{dv}{dz} + \left[\lambda_n - \mu(\mu+1)\right] v = 0 \ . \tag{8.33}$$

If we consider the expansion of v in a power series about $z = 0$, the requirement that the series converge everywhere leads to the result

$$\lambda_n = (\mu+\ell)(\mu+\ell+1) = (m\pi/\beta + \ell)(m\pi/\beta + \ell + 1) \tag{8.34}$$

where ℓ is an integer ≥ 0. Hence ν_n is of the form described above.

Now set $\Theta = \pi/2$ in (7.16) and consider $\hat{x}$ as varying along the straight edge. r then represents the distance of a point from the edge. It is easily seen from the proof of Theorem 7.3 that (7.16) converges uniformly with respect to r, ϕ and $\hat{x}$.* Next, in(7.16) let us represent each Bessel function by its power series expansion. By slightly modifying the proof of Theorem 7.3, it is easily shown that if every term in each series is replaced by its absolute value, (including the coefficients a_n and V_n), the resulting series is still convergent uniformly in r, ϕ and $\hat{x}$. Hence we can rearrange the terms of the double series according to increasing powers of r. Thus

$$u = \sum_{n=1}^{\infty} c_n(\phi,\hat{x})\, r^{\gamma_n} \tag{8.35}$$

where γ_n is of the form $m\pi/\beta + \ell$, ℓ and m being integers with $\ell \geq 0$, $m > 0$; the series is uniformly convergent in some neighborhood of an interior segment of the straight edge.

In particular, if $\beta = 2\pi$, (e.g. at a straight portion of the edge of an aperture in an "acoustically soft" screen), u can be expanded near the edge in a power series in $r^{1/2}$. Analogous results can be obtained, of course, for the other boundary conditions mentioned in the above hypotheses.

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*We confine ourselves to points $\hat{x}$ on the straight edge such that $\sigma(r_o,\hat{x})$ does not intersect $\dot{G}$ except in the given plane surface elements; (r_o is some sufficiently small number > 0).

Appendix I. Regularity of continuous solutions of the reduced wave equation.

It is well known that there are functions $u(x)$ such that $\frac{\partial^2 u}{\partial x_i^2}$, $i = 1,2,3$, exist in a domain G, and there satisfy the Laplace equation

$$\Delta u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2} = 0 \qquad \text{(I-1)}$$

but u is not a regular solution, i.e., of class $C^{(2)}(G)$; what is more, u is actually discontinuous at some point of G. In fact, if u is a continuous solution of (I-1) in G, then it must be of class $C^{(2)}(G)$ (and hence harmonic in G.)* We shall show that the corresponding statement for the reduced wave equation is a simple consequence of

Lemma I - 1. Let σ be the interior of a sphere and Σ its boundary. Suppose u and f belong to $C^{(o)}(\overline{\sigma})$ and satisfy in σ the equation

$$\Delta u = f \,. \qquad \text{(I-2)}$$

Then the function

$$v = u + \frac{1}{4\pi} \int_{\sigma} \frac{f(x')}{|x-x'|} \, dV' \qquad \text{(I-3)}$$

is harmonic in σ.

Proof: We can obviously assume without loss of generality that u and f are real; we can also assume that $u = 0$ on Σ, since otherwise we can subtract from u a function continuous in $\overline{\sigma}$ which is harmonic in σ and equal to u on Σ.

By Weierstrass's approximation theorem, there exists a sequence of polynomials f_n which approaches f uniformly in $\overline{\sigma}$. For each f_n let u_n be the unique continuous solution of the equation

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* See Petrovsky [25] p. 169 for example.

$$\Delta u_n = f_n \tag{I-4}$$

in σ, with the boundary condition on Σ

$$u_n = 0 \,. \tag{I-5}$$

The lemma is obviously true if the functions u and f are replaced by u_n and f_n respectively; namely, the function

$$v_n = u_n + \frac{1}{4\pi} \int_\sigma \frac{f_n(x')}{|x-x'|}\, dV' \tag{I-6}$$

is harmonic in σ. For $f_n(x)$ is certainly Hölder continuous in σ, and consequently, (Kellogg [10], p. 156)

$$\Delta \left(\frac{1}{4\pi} \int_\sigma \frac{f_n(x')}{|x-x'|}\, dV' \right) = - f_n(x) \tag{I-7}$$

Thus v_n is a continuous solution of Laplace's equation, and hence harmonic. Next, we have

$$\lim_{n\to\infty} \int_\sigma \frac{f_n(x')}{|x-x'|}\, dV' = \int_\sigma \frac{f(x')}{|x-x'|}\, dV' \tag{I-8}$$

uniformly for $x \in \bar{\sigma}$. Thus, for $s \in \Sigma$, $\lim_{n\to\infty} v_n(s)$ exists, the convergence being uniform with respect to s. It follows that the harmonic functions $v_n(x)$ tend uniformly to a harmonic function $\hat{v}(x)$, and, as a consequence of (I-8), the functions $u_n(x)$ uniformly approach some limit function $\hat{u}(x)$. We shall prove that $u(x) \equiv \hat{u}(x)$, and therefore $v(x) \equiv \hat{v}(x)$, which would prove the lemma.

Let $w_n = u_n - u$, $w = \hat{u} - u$. We have

$$\Delta w_n = f_n - f \tag{I-9}$$

where w, $w_n = 0$ on Σ. Suppose $w \neq 0$ at some point ξ of σ, say $w(\xi) = \alpha > 0$. (Otherwise we can consider the functions $- w_n$, $- w$.) Then for some integer q we have, by the uniform convergence of w_n and f_n respectively,

$$\left| w_q - w \right| < \frac{\alpha}{2} \tag{I-10}$$

$$\left| f_q - f \right| < \frac{\alpha}{d^2} \tag{I-11}$$

Here d is the diameter of σ. Then $w_q(\xi) > \frac{\alpha}{2}$, and hence w_q has a maximum $M > \frac{\alpha}{2}$ at some interior point of σ. Let us translate coordinates so that this point becomes the origin, and define

$$\eta(x) = w_q(x) + \frac{M}{2d^2}\left(x_1^2 + x_2^2 + x_3^2\right) \tag{I-12}$$

Then $\eta(o,o,o) = M$, and, for $s \in \Sigma$, $\eta(s) < \frac{M}{2}$. Hence η has a maximum somewhere on the interior of σ. But at all points of σ we have, using (I-9) and (I-11),

$$\Delta\eta = \Delta w_q + \frac{3M}{d^2} > - \frac{\alpha}{d^2} + \frac{3\alpha}{2d^2} > 0 \tag{I-13}$$

However, at a maximum, $\eta(x)$ cannot have a second derivative $\frac{\partial^2 \eta}{\partial x_i^2} > 0$. Thus the assumption $w \neq 0$ leads to a contradiction, and we have

$$w \equiv \hat{u} - u \equiv 0 \tag{I-14}$$

which completes the proof.

<u>Theorem I - 1.</u> If u is a continuous solution of the reduced wave equation (2.4) in a domain G, then $u \in C^{(2)}(G)$.

<u>Proof:</u> Let σ be an arbitrary sphere whose closure is in G. Upon applying the lemma to the functions u and $f = -k^2 u$ in $\bar{\sigma}$, we see that the function

$$v = u - \frac{1}{4\pi} \int_\sigma \frac{k^2 u(x')}{|x-x'|} \, dV' \tag{I-15}$$

is harmonic in σ. By Korn's result concerning volume distributions, Lemma 3.1, the continuity of u implies that the integral is in $C^{(1+\lambda)}(\sigma)$, whence, by the analyticity of v, $u \in C^{(1+\lambda)}(\sigma)$; this in turn implies that the integral is in $C^{(2+\lambda)}(\sigma)$, again by Korn's theorem. From the analyticity of v we now see that u is certainly in $C^{(2)}(\sigma)$ and the theorem is proved.

Appendix II. A property of regular closed surfaces.

We shall prove

Lemma 4.1. Let G be the exterior of a regular closed surface which is subdivided into a set $\mathcal{F}$ of $C^{(2+\lambda)}$ surface elements, and let $\mathcal{E}$ be the set of edge points of the elements of $\mathcal{F}$. Then there exists a number Θ, where $0 < \Theta \leq 1$, such that for $x \epsilon G$ the set $\tau_{\bar{G}}(\Theta d(x, \mathcal{E}), x)$ intersects at most one element of $\mathcal{F}$.

Proof: For a fixed $\hat{x} \epsilon \mathcal{E}$, let $F_1, F_2, \ldots, F_p$ be the elements of $\mathcal{F}$ containing $\hat{x}$, and let α_{ij}, $1 \leq i < j \leq p$, be the exterior angle at $\hat{x}$ formed by F_i and F_j. By Definition 5, $\alpha_{ij} \neq 0$.

Suppose that $\alpha_{ij} \neq \pi$ or 2π, for a given pair i, j. Then for some $\rho_{ij} > 0$ there exists a $C^{(2+\lambda)}$ coordinate transformation of the whole space, $x' = g_{ij}(x)$, which simultaneously flattens the sets* $F_i \cap \bar{\tau}_{\bar{G}}(\rho_{ij}, \hat{x})$ and $F_j \cap \bar{\tau}_{\bar{G}}(\rho_{ij}, \hat{x})$ the representative sets in the new coordinate system lying in mutually perpendicular planes. This coordinate transformation is obtained by forming a product of two transformations of the type (3.25)

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* These sets may not be surface elements, since they are not necessarily connected.

of Lemma 3.2. Assuming that F_i has the representation (2.2), we first flatten F_i by the transformation $x'' = g_1(x)$, wherein $x_1 = x_1''$ and $x_2 = x_2''$. Now F_j''* is not tangent to F_i'' at $\hat{x}''$. Hence there is a sphere $\bar{\sigma}(\rho'', \hat{x}'')$ in which F_j'' (or an extension of F_j'', if necessary) is represented by an equation giving either x_1'' or x_2'' in terms of the other two coordinates on a region $\bar{D}$, say,

$$x_1'' = h(x_2'', x_3''), \qquad (x_2'', x_3'') \in \bar{D}. \tag{II-1}$$

Here $h \in C^{(2+\lambda)}(\bar{D})$. (The existence of the indicated extension of F_j'' and the representation (II-1) in $\bar{\sigma}(\rho'', \hat{x}'')$ follows from Definition 3 and the implicit function theorem respectively.) We can then apply a transformation $x' = g_2(x'')$ which flattens $F_j'' \cap \sigma(\rho'', \hat{x}'')$; here we let the coordinate x_1'' play the part of x_3 in (3.25), so that F_i' will remain flat. Thus the transformation $g_{ij} = g_2 g_1$ has the desired properties. ρ_{ij} need only be taken so small that $g_1(\bar{\sigma}(\rho_{ij}, \hat{x})) \subset \sigma(\rho'', \hat{x}'')$.

We can now choose a $\hat{\rho} > 0$ satisfying the following conditions:

(a) $\hat{\rho}$ is less than the minimum distance from $\hat{x}$ to any member of $\mathcal{F}$ not containing $\hat{x}$;

(b) $\hat{\rho} \leq \rho_{ij}$ for any i,j such that $\alpha_{ij} \neq \pi$ or 2π.

(c) for any i, $1 \leqq i \leqq p$, the outgoing normal (from G) at every point of $\overset{\circ}{F}_i \cap \bar{\tau}_{\bar{G}}(\hat{\rho}, \hat{x})$ forms an angle with its limiting position at $\hat{x}$ which is less than $\pi/8$.

For $y \in \tau_G(\hat{\rho}/2, \hat{x})$, let ρ_y be the largest number for which $\tau_{\bar{G}}(\rho_y, y)$ intersects at most one member of $\mathcal{F}$; thus $\bar{\tau}_{\bar{G}}(\rho_y, y)$ intersects at least two elements of $\mathcal{F}$, which by (a) must contain $\hat{x}$, say F_1 and F_2. If $\bar{\tau}_{\bar{G}}(\rho_y, y) \cap \mathcal{E} \neq \emptyset$, then the conclusion of the lemma holds for this y with $\Theta = 1$.

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* As before, primes will be appended to letters to indicate the corresponding elements in the new coordinate system.

We shall suppose, therefore, that $\bar{\tau}_{\bar{G}}(\rho_y,y)$ contains only inner points of F_1 and F_2. Thus there are points z and w in $\bar{\tau}_{\bar{G}}(\rho_y,y)$ with $z \in \overset{\circ}{F}_1$, and $w \in \overset{\circ}{F}_2$ such that the line segments $\overline{yz}$ and $\overline{yw}$ are perpendicular to F_1 and F_2 respectively, and

$$\rho_y = \max(|y-z), |y-w|) . \tag{II-2}$$

It is then obvious, from condition (c) on $\hat{\rho}$, that $\alpha_{12} \neq 2\pi$. We now consider three cases according as both F_1 and F_2 are plane surface elements or not, and, under the latter alternative, $\alpha_{12} = \pi$ or not.

A. Both F_1 and F_2 are plane surface elements. In this case it is clear that $0 < \alpha_{12} < \pi$. Let ℓ be the line of intersection of the planes of F_1 and F_2, and let $r = d(y, \ell)$. (Except for $\hat{x}$, ℓ need not contain any points of $\mathfrak{F}$.) We see that

$$\rho_y \geq r \sin \frac{\alpha_{12}}{2} \tag{II-3}$$

Next we obtain an upper estimate for $d(y, \mathfrak{F})$ by considering the straight lines through z and w perpendicular to ℓ, meeting ℓ in the common point q. Either $q \in \mathfrak{F}$ or one of the segments $\overline{zq}$ or $\overline{wq}$ meets $\mathfrak{F}$ before reaching ℓ. In either case we have

$$d(y, \mathfrak{F}) < 2r \tag{II-4}$$

Hence

$$\rho_y \geq \frac{1}{2} \sin \frac{\alpha_{12}}{2} d(y, \mathfrak{F}) \tag{II-5}$$

Thus the conclusion of the lemma holds in this case with $\Theta = \frac{1}{2} \sin \frac{\alpha_{12}}{2}$.

B. F_1 and F_2 are not both plane, and $\alpha_{12} \neq \pi$. By condition (b) on $\hat{\rho}$, the coordinate transformation $x' = g_{ij}(x)$ flattens the sets $\hat{F}_i = F_i \cap \bar{\tau}_{\bar{G}}(\hat{\rho}, \hat{x})$, $i = 1,2$. Let $\hat{G} = E^3 - (\hat{F}_1 \cup \hat{F}_2)$, and let $\hat{\mathfrak{F}} = e(\hat{F}_1) \cup e(\hat{F}_2)$. Here $e(\hat{F}_i)$ is defined as follows: for any admissible coordinate system, let $\hat{F}_i$ have the representation

$$x_3 = f_i(x_1, x_2) , \qquad (x_1, x_2) \in S_i ; \tag{II-6}$$

then $e(\hat{F}_i)$ is the set of points $(x_1, x_2, f_i(x_1, x_2))$ where $(x_1, x_2, f_i(x_1, x_2)) \in \dot{S}_i$. We now apply the reasoning used in (A) and in the paragraph preceding (A), with $\hat{G}'$ and $\hat{\mathfrak{F}}'$ in place of $\tau_{\bar{G}}(\rho/2, x)$ and $\mathfrak{F}$ respectively*. Since the plane of $\hat{F}_1'$ is perpendicular to that of $\hat{F}_2'$, then for all $x' \in \hat{G}'$, the set $\tau_{\hat{G}'}((\sqrt{2}/4) d(x', \hat{\mathfrak{F}}'), x')$ does not intersect both $\hat{F}_1'$ and $\hat{F}_2'$. In particular the set $\tau_o = \tau_{\hat{G}'}((\sqrt{2}/4) d(y', \hat{\mathfrak{F}}'), y')$ fails to intersect one of these sets, say $\hat{F}_1'$. Hence the set $g^{-1}(\tau_o)$ fails to intersect $\hat{F}_1$. (It could possibly intersect $F_1 - \hat{F}_1$, as well as F_2, or other surface elements of $\mathcal{F}$.) But by Lemma 3.2(f), there is a constant C_1 depending only on the transformation g_{12} such that

$$d(y, g^{-1}(\dot{\tau}_o - \hat{F}_2')) \geq C_1^{-1} \frac{\sqrt{2}}{4} d(y', \hat{\mathfrak{F}}') \geq C_1^{-2} \frac{\sqrt{2}}{4} d(y, \hat{\mathfrak{F}}). \tag{II-7}$$

Since $d(y, \mathfrak{F}) \leq |y - \hat{x}| < \hat{\rho}/2$, we easily see that

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* I.e. we temporarily ignore $\dot{G}'$ except for $\hat{F}_1' \cup \hat{F}_2'$.

$$d(y,\hat{\mathcal{P}}) \geq d(y,\mathcal{P}) \tag{II-8}$$

Upon setting $\rho = d(y,g^{-1}(\dot{\tau}_o-\hat{F}_2'))$, we see that $\tau_{\bar{G}}(\rho,y)$ does not intersect $\hat{F}_1$. Thus $\rho \leq \rho_y$, and we have from (II-7) and (II-8)

$$\rho_y \geqq \frac{\sqrt{2}}{4} C_1^{-2} d(y,\mathcal{P}) \tag{II-9}$$

Thus the conclusion of the lemma holds in this case with $\Theta = \frac{\sqrt{2}}{4} C_1^{-2}$.

C. $\alpha_{12} = \pi$. We can assume that the outgoing normal at z is parallel to the x_3-axis, and is in the direction of decreasing x_3. Thus y and z have the same x_1, x_2 coordinates, with $y_3 > z_3$.* Moreover, from condition (c) on $\hat{\mathcal{P}}$, it is easily seen that $y_3 > w_3$. It also follows from (c) that the sets $\hat{F}_i$ have the representation (II-6) for i=1 and 2, where

$$\left(\frac{\partial f_i}{\partial x_1}\right)^2 + \left(\frac{\partial f_i}{\partial x_2}\right)^2 < 1, \quad (x_1,x_2) \in S_i \ , \quad i = 1,2 \ . \tag{II-10}$$

In order to estimate $d(y,\mathcal{P})$ in terms of ρ_y, we note first that in the rectilinear triangle with vertices y, z, and w, the angle at y is less than $\pi/4$, by condition (c). Then by (II-2) we have

$$|z-w| < \rho_y \tag{II-11}$$

Now let us denote the points (z_1,z_2) and (w_1,w_2) in E^2 by s and t respectively. The line segment $\overline{st}$ cannot lie in $\mathring{S}_1 \cap \mathring{S}_2$. For, if it did,

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* We denote the coordinates of a point by appending subscripts to the letter designating it; thus y has the coördinates y_1, y_2, y_3.

we would have $f_1(w_1,w_2) < f_2(w_1,w_2)$, since $y_3 > f_1(y_1,y_2)$, and $\overline{yz}$ does not intersect F_1; moreover $f_1(z_1,z_2) > f_2(z_1,z_2)$: but this is impossible since there would then be a point common to $\overset{\circ}{F}_1$ and $\overset{\circ}{F}_2$. Thus there exists on $\overline{st}$ either a point $s_1 = (\tilde{z}_1,\tilde{z}_2)$ of $\dot{S}_1$ or a point $t_1 = (\tilde{w}_1,\tilde{w}_2)$ of $\dot{S}_2$. In the first case we can further suppose that s_1 is the point in $\overline{st} \cap \dot{S}_1$ nearest to s; then the curve represented by

$$x_3 = f_1(x_1,x_2) \ , \quad (x_1,x_2) \in \overline{ss_1} \tag{II-12}$$

constitutes a path from z to $\hat{\xi}$ (see case B) of length less than $\sqrt{2}|s-s_1|$, by (II-10). Hence $d(z,\hat{\xi}) \leq \sqrt{2}|z-w|$. A similar argument in the second case shows that $d(w,\hat{\xi}) \leq \sqrt{2}|z-w|$. In either case we have, by the triangle inequality,

$$d(y,\hat{\xi}) \leq \rho_y + \sqrt{2}|z-w| \tag{II-13}$$

As in case B, $\hat{\xi}$ has the property (II-8); this, together with (II-11) and (II-13) gives us

$$\rho_y \geq \frac{1}{3}\, d(x,\xi) \ . \tag{II-14}$$

Thus the conclusion of the lemma holds in this case with $\Theta = \frac{1}{3}$.

For the given $\hat{x} \in \xi$ there are only a finite number of angles α_{ij} and a finite number of flattening transformations g_{ij} with their associated constants C_1, so that there exists a $\Theta = \Theta(\hat{x}) > 0$ such that

$\theta \leq \frac{1}{3}$, $\theta < \frac{1}{2} \sin \frac{\alpha_{ij}}{2}$, and $\theta < \frac{\sqrt{2}}{4} C_1^{-2}$ for all i,j, $1 \leq i < j \leq p$. Thus the lemma has been shown to hold for all x in the neighborhood $\tau_{\bar{G}}(\hat{\delta}/2,\hat{x})$, in the relative topology of $\bar{G}$, for this value of θ. Next for all $x \in \bar{G} - \mathfrak{F}$ there exists a neighborhood of x and a $\theta = \theta(x)$ with the same property. Finally, the lemma holds for all x in the exterior of a sufficiently large sphere $\sigma(\rho_1,0)$ containing $\dot{G}$ in its interior, with $\theta = \frac{1}{2}$. The lemma then follows easily from the compactness of the set $\bar{\sigma}(\rho_1,0) \cap \bar{G}$.

Appendix III. The eigenvalue problem for the Beltrami operator in spherical surface regions.

The methods used in Section 7 to extend Rellich's growth estimate to eventually conical domains and the uniqueness theorem to boundaries with conical points depend on the following results.

We consider domains Ω on the surface of the unit sphere with boundaries consisting of a finite number of simple $C^{(2+\lambda)}$ arcs,** any two of which have at most endpoints in common; we assume the interior angle between two arcs with a common endpoint is never zero. To each arc we assign one of the boundary conditions $V=0$ or $\partial V/\partial n = 0$. We make the restriction that if Ω is multiply connected, then the same boundary condition is assigned to all of $\partial\Omega$, with the possible exception of one connected component of $\partial\Omega$; on some parts, but not all, of the latter component, the other boundary condition may hold. If

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** These are defined in Section 7. See footnote (**), p.46.

we define the boundary operator $\mathcal{B}$ at every point of $\partial\Omega$ as equal to the identity operator, I, or to $\partial/\partial n$, according as the first or second boundary condition holds, we can write the boundary condition as $\mathcal{B}V = 0$.

Under these conditions, we have

Theorem III-1. There exists a complete orthonormal set of eigenfunctions $V_n(\omega)$ for the operator $-\Delta^*$ on Ω with the prescribed boundary conditions. Thus the functions $V_n(\omega)$ are continuous in $\bar{\Omega}$ and, together with their associated eigenvalues λ_n, satisfy

$$\Delta^* V_n + \lambda_n V_n = 0 \qquad \omega \in \Omega \qquad \text{(III-1)}$$

$$\mathcal{B} V_n = 0 \qquad \omega \in \partial\Omega \qquad \text{(III-2)}$$

Moreover, V_n and λ_n possess the following additional properties:

(a) $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$, and

$$\sum_{n=1}^{\infty} 1/\lambda_r^2 < \infty \quad ; \qquad \text{(III-3)}$$

(b) there exists a constant $Q > 0$ such that for all $\omega \in \Omega$, and all n for which $\lambda_n > 0$,

$$|V_n| \leq Q\lambda_n^2 \qquad \text{(III-4)}$$

Proof: Let us assume, for definiteness, that $\mathcal{B} = I$ on all but one component Γ of $\partial\Omega$, on which mixed boundary conditions obtain. The proof is not essentially different if the boundary conditions in this situation are interchanged or if none of the components have mixed boundary conditions; any non-trivial changes required for these cases will be indicated later. The proof is divided into five parts as follows:

(A) the transformation of the eigenvalue problem, by a conformal mapping, to one on the plane involving the ordinary Laplacian, in a region whose exterior boundary is a semicircle;

(B) the reduction of the mixed boundary value problem to a Dirichlet boundary problem by a reflection;

(C) the transformation of the Dirichlet problem to one with circular boundaries, by another conformal mapping, and thence to an integral equation.

(D) the derivation of an orthonormal set of eigenfunctions and some of their properties from the integral equation;

(E) the positiveness of the eigenvalues.

A. Choosing any point ω_o not on $\partial\Omega$ as the north pole, we transform the eigenvalue problem by a stereographic projection T of the sphere onto the tangent plane at the south pole. As usual, we represent the points of the plane by the complex numbers, the point of tangency corresponding to zero; the mapping can thus be written as $z = x + iy = T(\omega)$. If $\omega_o \in \Omega$, then the image Ω' of Ω is unbounded, containing the point ∞, but this does not cause any difficulty in what follows.** Under this mapping, the operator Δ^* is transformed into $(1+r^2/4)^2\Delta$, where Δ is the ordinary two-dimensional Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$, and r is the distance of the image point $T(\omega)$ from the origin (the point of tangency). The element of area $d\omega$ on the sphere is transformed into $(1+r^2/4)^{-2}dxdy$.

Next, there exists a conformal mapping $\zeta = f(z)$ which maps that domain bounded by $\Gamma' = T(\Gamma)$ which contains Ω' onto the upper half of the unit circle in the ζ-plane. $f(z)$ can also be so chosen that the circular part of $\Gamma'' = f(\Gamma')$ corresponds to one or more adjoining arcs of Γ on which $\mathcal{B} = I$, and the corners

** It is not always possible to take ω_o as an exterior point of Ω, since $\partial\Omega$ may consist only of slits on the sphere.

$\zeta = \pm 1$ each correspond to endpoints of arcs (possible the same arc) on which $\mathcal{B} = \partial/\partial n$. Under the mapping $\zeta = \xi + i\eta = f(z)$, the Laplacian in the z-plane is transformed into $|f'(z)|^2\Delta_\zeta$, where $\Delta_\zeta = \partial^2\partial\xi^2 + \partial^2/\partial\eta^2$; the Jacobian is $\frac{\partial(\xi,\eta)}{\partial(x,y)} = |f'(z)|^2$. Equation (III-1) becomes

$$\Delta_\zeta V_n + \lambda_n \rho(\zeta) V_n = 0, \qquad \text{(III-5)}$$

where

$$\rho(\zeta) = \left[|f'(z)|(1+r^2/4)\right]^{-2} . \qquad \text{(III-6)}$$

The boundary condition at every point of $\partial\Omega''$ is, of course, the same as that given by (III-2) for the corresponding point of $\partial\Omega$, and we shall use the same symbol for the transformed boundary operator.

B. We now convert the mixed boundary value problem to a Dirichlet problem as follows. We reflect the domain Ω'' in the real axis, $\eta = 0$, and remove those parts of the boundary on the line $\eta=0$ on which $\mathcal{B} = \partial/\partial n$. Ω'' thus joins with its reflection to form a domain Ω_2''. This domain is a circle with, possibly, a number of "holes"; these consist of the original "holes", if any, of Ω'', together with their reflection in the ξ-axis, as well as slits on the ξ-axis corresponding to those sections on the straight part of Γ'' on which $\mathcal{B} = I$. Next, we define $\rho(\zeta)$ in the lower half of Ω_2'' to be an even function of η; i.e. we set

$$\rho(\bar{\zeta}) = \rho(\zeta) . \qquad \text{(III-7)}$$

$\rho(\zeta)$ is continuous at every interior point of Ω_2'', and has continuous first derivatives except at those points of Ω_2'' lying on the line $\eta=0$; at such points the derivatives have at most simple discontinuities. With respect to points on $\partial\Omega_2''$ the same is true except possibly at a finite number of points corresponding to corners of Γ', or at $\zeta = \pm 1$, where ρ may become infinite.

Now let us suppose that we have established the existence of a complete orthonormal set of continuous eigenfunctions $U_n(\zeta)$ for (III-5) in the domain Ω_2'', satisfying the condition $U_n=0$ on $\partial\Omega_2''$. Here the terms "orthogonal", "norm", and "completeness" refer to the inner product

$$(\phi_1,\phi_2) = \iint_{\Omega_2''} \phi_1(\zeta)\phi_2(\zeta)\rho(\zeta)d\xi d\eta \ . \tag{III-8}$$

We suppose, also, that the eigenvalues $\tilde{\lambda}_n$ associated with $U_n(\zeta)$ are real and satisfy (III-3), and that, for some constant $Q_1 > 0$

$$|U_n| \leq Q_1|\tilde{\lambda}_n| \qquad \zeta \in \Omega_2'' , \ n = 1,2,\ldots \tag{III-9}$$

It follows easily from (III-7) that for each n the function $U_n(\bar{\zeta})$ also satisfies (III-5) with the same eigenvalue $\tilde{\lambda}_n$. We then see that (III-5) and the boundary condition U=0 on $\partial\Omega_2''$ are also satisfied by the functions $U_n^e(\zeta) = U_n(\zeta) + U_n(\bar{\zeta})$, and $U_n^o(\zeta) = U_n(\zeta) - U_n(\bar{\zeta})$, which are even and odd respectively in η. Next, $U_n(\zeta) = 1/2[U_n^e(\zeta) + U_n^o(\zeta)]$ so that the functions $U_n^e(\zeta)$, $U_n^{(o)}(\zeta)$ also constitute a complete set on Ω_2''(in general not normalized or orthogonal). Since any continuous function on Ω'' can be extended continuously as an even function on Ω_2'', it is easy to see that the even functions $U_n^e(\zeta)$ form a complete set on

Ω''. Moreover, U_n^e has a vanishing normal derivative on those sections of $\partial\Omega''$ on which $\mathcal{B} = \partial/\partial n$. The set U_n^e can then be orthonormalized, yielding the set of functions V_n, complete and orthonormal on Ω'', and satisfying the boundary condition $\mathcal{B} V_n = 0$. (Here, of course, we use the inner product obtained from (III-8) by replacing Ω_2'' by Ω''.)

We now show, using (III-9), that the functions $V_n(\zeta)$ and their associated eigenvalues λ_n * satisfy (III-4). Each V_j is a linear combination of those functions $U_{p+1}, U_{p+2}, \ldots, U_{p+m}$ of the postulated orthonormal set on Ω_2'' which belong to the eigenvalue λ_j. Let us again consider V_n in the entire domain Ω_2''; it is, of course, even in η, and its norm, according to (III-8), is $\sqrt{2}$. Then we have, for the normalized function,

$$2^{-1/2} V_j(\zeta) = \frac{\sum\limits_{i=1} \alpha_i U_{p+i}}{\left(\sum\limits_{i=1}^{m} |\alpha_i|^2 \right)^{1/2}} \tag{III-10}$$

To this we apply (III-9), and a special case of Cauchy's inequality

$$\sum_{i=1}^{m} |\alpha_i| \leq \sqrt{m} \left(\sum_{i=1}^{m} |\alpha_i|^2 \right)^{1/2} . \tag{III-11}$$

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* We renumber the eigenvalues so that their indices correspond to their order with respect to the functions V_n. Some of the original eigenvalues may have only odd eigenfunctions U_n associated with them, and would consequently be dropped.

We obtain

$$|V_j(\zeta)| \leqq \sqrt{2m}\, Q_1 |\tilde{\lambda}_j| \tag{III-12}$$

But if $\mu(\tilde{\lambda}_n)$ denotes the multiplicity of the eigenvalue $\tilde{\lambda}_n$, then by (III-3) we must have for all sufficiently large n

$$\mu(\tilde{\lambda}_n) \leqq \tilde{\lambda}_n^2 \tag{III-13}$$

Hence there exists a constant C such that for all n

$$\mu(\tilde{\lambda}_n) \leqq C\tilde{\lambda}_n^2 \tag{III-14}$$

Thus $m \leqq C\lambda_j^2$; using this in (III-12), and setting $Q = \sqrt{2C}\, Q_1$, we obtain (III-4).

Upon applying the inverse transformations $f^{-1}(\zeta)$ and $T^{-1}(z)$ to (III-5) and to the integral (III-8) with Ω_2'' replaced by Ω'', recalling how the element of area is transformed in each case, we obtain the results stated in the theorem.

C. It remains for us to prove the existence and required properties of the complete orthonormal set of eigenfunctions $U_n(\zeta)$ for the domain Ω_2''. To do this we apply still another conformal transformation, $w = s + it = h(\zeta)$, which maps Ω_2'' onto a finite domain Ω_2 whose boundary consists entirely of circles. (The existence of such a transformation is proved in [26].) For this transformation the Jacobian $\frac{\partial(s,t)}{\partial(\zeta,\eta)} = |h'(\zeta)|^2$, equation (III-5) becomes

$$\Delta_w U_n + \lambda_n \tilde{\rho}(w) U_n = 0 \tag{III-15}$$

where

$$\tilde{\rho}(w) = |h'(\zeta)|^{-2} \rho(\zeta)$$

$\tilde{\rho}(w)$ is continuous in the interior of Ω_2 with continuous derivatives, except possibly on a finite number of curves on which a derivative may have a simple discontinuity. These curves correspond to the line segments on which $\rho(\zeta)$ has the same property. Finally, $\tilde{\rho}(w)$ may become infinite at a finite number of boundary points of Ω_2, corresponding to corners of $\partial\Omega'$ or to $\zeta = \pm 1$

Let $G(w,\upsilon)$, $(\upsilon = \sigma + it)$, be the Green's function for the operator $-\Delta_w$ in the domain Ω_2^*. We have

$$G(w,\upsilon) = -\frac{1}{2\pi}\log|w-\upsilon| + g(w,\upsilon) \tag{III-17}$$

where g is harmonic with respect to w in Ω_2 for each υ in $\bar{\Omega}_2$, and vice versa. It is not difficult to show that $g \in C^{(0)}(\bar{\Omega}_2 \times \bar{\Omega}_2 - [\mathcal{J} \cap (\partial\Omega_2 \times \partial\Omega_2)])$ where $\mathcal{J} = \{(w,\upsilon): w=\upsilon\}$; moreover, due to the smoothness of the boundaries, $g \in C^{(2)}(\Omega_2 \times \bar{\Omega}_2)$.

We now consider the related homogeneous integral equation

$$U_n(w) = \tilde{\lambda}_n \iint_{\Omega_2} G(w,\upsilon)U_n(\upsilon)\rho(\upsilon)d\sigma d\tau \tag{III-18}$$

whose solutions must satisfy (III-15), as well as the boundary condition $U_n = 0$ on $\partial\Omega_2$. The proof of this follows the one given in [23] for the case where $\tilde{\rho}$ is continuous in $\bar{\Omega}_2$; the finite number of discontinuities of $\tilde{\rho}$ on the boundary, and the finite number of lines on which its derivatives are discontinuous,

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* The existence of the Green's function for a region of this type is proved in [26].

do not involve any radical changes in the proof.*

D. We now apply to (III-18) a slight generalization of the theory of integral equations with symmetric kernels developed in [23].** The generalization arises if we regard the integral in (III-18) as the integral of $G(w,\upsilon)U_n(\upsilon)$ with respect to the measure μ defined by

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* As in the proof of [23], (pp. 366-368), if $U_n \epsilon C^{(o)}(\bar{\Omega}_2)$, the integral

$$X(s,t) = \iint_{\Omega_2} g(w,\upsilon)U_n(\upsilon)\tilde{\rho}(\upsilon)d\sigma d\tau$$

and the integrals obtained from it by replacing g by g_s, g_{ss}, etc. converge uniformly for $w \epsilon \bar{\Omega}_o \subset \Omega_2$. This follows from the smoothness properties of g stated above. Thus we can compute $\Delta_w X$ by differentiating under the integral sign. The evaluation of

$$\psi = \iint_{\Omega_2} \log|w-\upsilon|U_n(\upsilon)\tilde{\rho}(\upsilon)d\sigma d\tau$$

presents no problem. We can avoid some difficulty at the boundary points where ρ is infinite if we divide the region of integration into two parts as follows: one part consists of a closed subregion containing a neighborhood of the point (s,t), and contained in the open region Ω_2, the integral over this part being labeled as ψ in [23]; the integral over the remaining part is treated as X(s,t), above.

The relationship between (III-15) and (III-18) is actually an equivalance; i.e. we can also show that any solution of (III-15) which satisfies the boundary condition must satisfy (III-18). However, the analysis involves some new complications connected with the behavior of $\partial U_n/\partial n$ near a boundary point where ρ is infinite, and, since we do not need the implication in this direction, we omit the proof.

** It should be noted that the device generally used to convert an equation like (III-18) to one with a symmetric kernel is not suitable here. This device consists of multiplying (III-18) by $\tilde{\rho}(w)$, introducing the new symmetric kernel $\mathcal{Y}(w,\upsilon) = G(w,\upsilon)\sqrt{\tilde{\rho}(w)}\sqrt{\tilde{\rho}(\upsilon)}$, and then considering the eigenfunctions $u_n = \sqrt{\tilde{\rho}(w)}\,U_n(w)$. In our case, however, $\sqrt{\tilde{\rho}(w)}$ may be infinite at certain points, and hence the kernel $\mathcal{Y}$ violates the requirement ([23], p. 152) that $\iint_{\Omega_2} [\mathcal{Y}(w,\upsilon)]^2 d\sigma dt$ be uniformly bounded with respect to w.

$$d\mu = \tilde{\rho}(\upsilon)d\sigma d\tau \tag{III-19}$$

Throughout the theory of symmetric integral equations presented in Chapter III, §§ 1,2,4,9, of [23], we replace each integral by one with respect to this measure. Thus the inner product becomes

$$(\phi_1,\phi_2) = \iint_{\Omega_2} \phi_1(w)\phi_2(w)\tilde{\rho}(w)dsdt \tag{III-20}$$

Similarly we consider the "quadratic integral form"

$$J(\phi,\phi) = \iint_{\Omega_2} \iint_{\Omega_2} G(w,\upsilon)\phi(w)\phi(\upsilon)\tilde{\rho}(w)\tilde{\rho}(\upsilon)d\sigma d\tau dsdt \tag{III-21}$$

and so forth.

All the methods and results of the theory in the sections of [23] indicated above are valid for this generalization. In particular, the extension to the case where the kernel may become infinite is still valid. As in the regular theory, we require that there exist a constant C such that

$$\iint_{\Omega_2} [G(w,\upsilon)]^2\tilde{\rho}(\upsilon)d\sigma d\tau \leq C , \qquad w \in \bar{\Omega}_2 \tag{III-22}$$

To prove this, we note that G has only logarithmic singularities, and that near a boundary point w_o where ρ becomes infinite, we must have

$$\tilde{\rho}(w) \sim A|w-w_o|^{-2+\delta} , \qquad \delta > 0 , \; A > 0 \tag{III-23}$$

The proof of (III-23) depends on the fact that the interior angles between adjacent $C^{(2+\lambda)}$ arcs of $\partial\Omega$ do not vanish. It is an immediate consequence of Lemma III-1, given below.

The existence of a complete orthonormal set of eigenfunctions $U_n(w)$ with associated real eigenvalues $\tilde{\lambda}_n$, satisfying (III-18), and hence (III-15), now follows. In addition, we have the inequality

$$\sum_{n=1}^{\infty} \frac{[U_n(w)]^2}{\tilde{\lambda}_n^2} \leq \iint_{\Omega_2} [G(w,\upsilon)]^2 \tilde{\rho}(\upsilon) d\sigma d\tau \tag{III-24}$$

From this we immediately get (III-9) for some Q_1. Next, integrating (III-24), we get

$$\sum_{n=1}^{\infty} \frac{1}{\tilde{\lambda}_n^2} \leq \iint_{\Omega_2} \iint_{\Omega_2} [G(w,\upsilon)]^2 \tilde{\rho}(\upsilon)\tilde{\rho}(w) d\sigma d\tau ds dt \tag{III-25}$$

which gives us (III-3).

E. The positiveness of the eigenvalue λ_n is proved by referring to the original equation (III-1) on the sphere. As usual we apply the self adjointness property of $-\Delta^*$, i.e. Green's theorem on the sphere:** if $u,v \in C^{(1)}(\bar{\Omega})$, and $u \in C^{(2)}(\Omega)$, then

$$\int_{\Omega} v\Delta^* u \, dw + \int_{\Omega} \nabla^* v \cdot \nabla^* u \, dw = \int_{\partial\Omega} \frac{\partial u}{\partial n} ds \tag{III-26}$$

Here ∇^* is the gradient on the surface of the sphere; in polar coordinates we have

$$\nabla^* = \left(\frac{\partial}{\partial \Theta} , \frac{1}{\sin \Theta} \frac{\partial}{\partial \phi} \right) \tag{III-27}$$

** Cf. the proof of Lemma 7.1.

Upon setting $v = u = V_n$ in (III-26), and substituting for $\overset{*}{\Delta} V_n$, using (III-1), we get the required result. As in the three dimensional case where edges are involved, there are complications near corners of Ω. In general V_n does not satisfy the smoothness conditions presupposed in (III-26); one or more derivatives of V_n may become infinite at a corner. However, in this case we can still establish (III-26) by surrounding each corner with a circle of arbitrary small radius ϵ, applying (III-26) to the remaining region and then letting $\epsilon \to 0$. As in the three-dimensional case, the proof depends on the fact that $\epsilon\, \partial V_n/\partial n \to 0$ uniformly as $\epsilon \to 0$; this is demonstrated in the proof of Lemma 7.1.

If the boundary conditions on $\partial\Omega$ assumed at the beginning of the proof were interchanged, so that $\mathcal{B} = \partial/\partial n$ on all but one component of $\partial\Omega$, the following changes in the proof should be noted:

(a) the hemispherical boundary Γ'' must be arranged so that $\mathcal{B} = \partial/\partial n$ on the circular part of Γ''; upon reflecting Ω'', we remove the parts of Γ'' on which $\mathcal{B} = I$; we still define ρ to be an even function in Ω'', but in constructing the eigenfunctions for Ω'' from the set $U_n(\zeta)$ on Ω_2'', we use odd eigenfunctions;

(b) in the resulting eigenvalue problem on the w-plane, with $\partial u/\partial n = 0$ on $\partial\Omega_2$, there is no Green's function since (III-14) has zero as an eigenvalue; however we can use a Neumann function, which is a "Green's function in the generalized sense", in essentially the same manner, to obtain the desired results.

We now prove a function-theoretic result which enables us to establish (III-23).

Lemma III-1. Let z_o be a boundary point of a domain D in the complex plane and suppose that in some neighborhood of z_o, ∂D consists of two $C^{(2+\lambda)}$ arcs Γ_1 and Γ_2 having z_o as a common endpoint. Let the interior angle α at z_o satisfy the condition $0 < \alpha \leq 2\pi$. Consider a function f(z), defined on $\overline{D}$, which maps D conformally onto the domain D', and which maps $\Gamma_1 \cup \Gamma_2$ onto a $C^{(2+\lambda)}$ arc Γ'.

Then there exists a constant $b \neq 0$ such that, as $z \to z_o$, $z \in \overline{D}$,

$$f(z) - f(z_o) \sim b(z - z_o)^{\pi/\alpha}, \tag{III-28}$$

$$f'(z) \sim \frac{\pi b}{\alpha} (z - z_o)^{\pi/\alpha - 1} \tag{III-29}$$

Proof: We show first that if $\hat{z}$ is any point of Γ_1(or Γ_2), other than an endpoint, then f'(z) exists* and is continuous at $\hat{z}$; moreover $f'(\hat{z}) \neq 0$.

Let D_1 be a simply connected subdomain of D with a piecewise smooth boundary which includes the subarc $\tilde{\Gamma}_1$ of Γ_1 contained in some neighborhood of $\hat{z}$. Because of the smoothness of Γ_1 there exists an open circle γ contained in D_1 and tangent to $\tilde{\Gamma}_1$ at $\hat{z}$; let z_c be the center of γ. Let $\zeta_1 = p(z)$ be a conformal mapping of D_1 onto the unit circle in the ζ_1-plane which maps z_c into the origin. Then $G(z) = -\log |p(z)|$ is the Green's function for D_1 with pole at z_c. Let H(z) be the function conjugate to G. By Theorem 3.1,

---

*Here we mean that $f'(\hat{z})$ exists as a "one-sided derivative", i.e. $\displaystyle\lim_{\substack{z \to \hat{z} \\ z \in D}} \frac{f(z) - f(\hat{z})}{z - \hat{z}}$ exists and is finite.

$G(z) \in C^{(2+\lambda)}(D_1 \cup \tilde{\Gamma}_1 - z_c)$;* consequently $H(z) \in C^{(2+\lambda)}(N_{\hat{z}})$, where $N_{\hat{z}}$ is some neighborhood of $\hat{z}$, relative to $\bar{D}_1$. It follows that the derivative of the analytic function $G(z) + iH(z)$ exists and is continuous at $\hat{z}$; hence the same is true of

$$p(z) = e^{-[G(z) + iH(z)]} \tag{III-30}$$

Next, the function $\log|z - z_c|$ has a minimum on ∂D_1 at $\hat{z}$. Therefore, the function $g(z)$ which is harmonic in D_1 and assumes boundary values equal to $\log |z - z_o|$ on ∂D_1 has a minimum at $\hat{z}$, and hence has an outgoing normal derivative $\partial g/\partial n \leq 0$ at $\hat{z}$. (The existence of $\partial g/\partial n$ follows from Theorem 3.1). But

$$G(z) \quad = \quad - \log |z - z_c| + g(z), \tag{III-31}$$

so that $\partial G/\partial n < 0$. Hence, by (III-30), we see that $p'(\hat{z}) \neq 0$.

Similarly, $D_1' = f(D_1)$ is mapped conformally onto the unit circle in the ζ_2-plane by an analytic function $\zeta_2 = q(w)$ such that $q'(w)$ is continuous at $\hat{w}$ and is not zero there ($\hat{w} = f(\hat{z})$). Now, let $\zeta_2 = L(\zeta_1)$ be the linear fractional transformation which maps the unit circle in the ζ_1-plane onto that of the ζ_2-plane, mapping $p(z)$ into $q^{-1}[f(z)]$. $L'(z)$ never vanishes, and hence the derivative of $f(z) = q^{-1}(L[p(z)])$ is continuous at $\hat{z}$ and does not vanish there.

Next, we choose a simply connected subdomain D_o of D such that

---

*Theorem 3.1 is obviously applicable in two dimensions: $G(z)$ can be regarded as a harmonic function in three variables which is constant with respect to the third variable.

(a) $\overline{D}_o$ includes the corner at z_o, i.e. ∂D_o includes subarcs of Γ_1 and Γ_2 terminating at z_o,

(b) D_o is contained in a circle γ_o with center at z_o, and radius so small that γ_o intersects ∂D only in Γ_1 and Γ_2, and the radius vector from z_o to any point of $\Gamma_1 \cup \gamma_o$, $(\Gamma_2 \cup \gamma_o)$, makes an angle $< \alpha/2$ with the tangent to Γ_1, (Γ_2), at z_o.

(c) ∂D_o consists of a $C^{(2+\lambda)}$ arc beginning and ending at the corner z_o.

From the hypothesis concerning $f(\Gamma_1 \cup \Gamma_2)$ and the fact that $f'(z) \neq 0$ at every point of ∂D_o, except possibly at z_o, we see that $D_o' = f(D_o)$ has a boundary consisting of a closed $C^{(2+\lambda)}$ curve (cf. footnote p. 46).

Let D_o^v be the image of D_o under the transformation $v = (z - z_o)^{\pi/\alpha}$. It is not difficult to show that ∂D_o^v is a closed $C^{(1+\lambda)}$ curve. Now consider the transformation

$$w = X(v) = f(z_o + v^{\alpha/\pi}) \tag{III-32}$$

mapping D_o^v onto D_o'. We apply to X the same reasoning as that used above for the mapping f, with one important change, to show that $X'(v)$ exists, is continuous, and does not vanish at $v = 0$.

Let $\zeta_1 = p_o(v)$ be a function which maps D_o^v conformally onto the unit circle in the ζ_1-plane. By [30], (Theorems I and IV), the derivatives of the Green's function $-\log |p_o(v)|$ exist and are continuous in $\overline{D}_o^v$, and do not both vanish at the boundary.* The rest of the above reasoning in connection with $f'(z)$ now applies, and yields the desired result concerning $X'(v)$. (III-28) and (III-29) then follow immediately from

$$f(z) - f(z_o) = X[(z - z_o)^{\pi/\alpha}] - X(0). \tag{III-33}$$

*The method used to prove the corresponding result for the mapping f at $\hat{z}$ is not suitable in the present case; in the former situation use was made of the fact that a circle could be drawn lying in D_1 and tangent to the boundary at $\hat{z}$. This is not always possible when the boundary is only known to be a closed $C^{(1+\lambda)}$ curve.

ACKNOWLEDGMENT

The author wishes to thank the members of the faculty and staff of the Institute of Mathematical Sciences for their many kindnesses. In particular, he would like to express his appreciation to Professor Joseph B. Keller for his guidance and help in the preparation of this paper.

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